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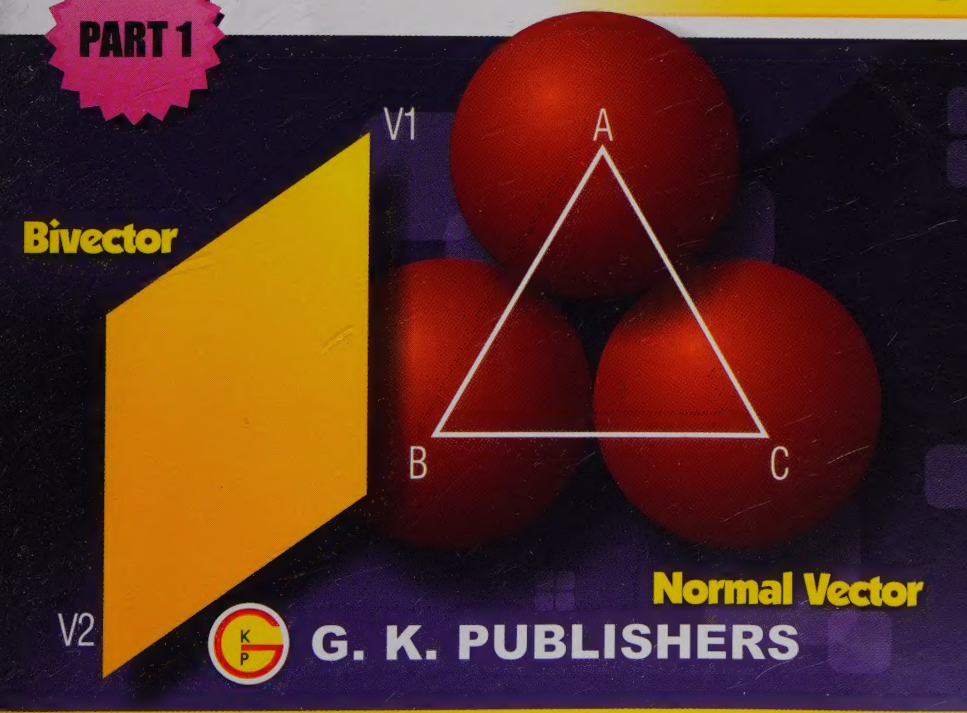
PLANE
Trigonometry

S. L. Loney

PART 1



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PLANE TRIGONOMETRY

By

S.L. LONEY

(Metric Edition)

PART I

An Elemetrary Course,
Excluding the use of Imaginary Quantities

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PREFACE

The following work will, I hope, be found to be a fairly complete elementary text-book on Plane Trigonometry, suitable for Schools and the Pass and Junior Honour Classes of Universities. In the higher portion of the book I have endeavoured to present to the students, as simply as possible, the modern treatment of complex quantities, and I hope it will be found that he will have little to unlearn when he commences to read treatises of a more difficult character.

As Trigonometry consists largely of formulae and the applications thereof, I have prefixed a list of the principal formulae which the student should commit to memory. These more important formulae are distinguished in the text by the use of thick type. Other formulae are subsidiary and of less importance.

The number of examples is very large. A selection only should be solved by the student on a first reading.

On a first reading also the articles marked with an asterisk should be omitted.

Considerable atention has been paid to the printing of the book and I am under great obligation to the Syndics of the Press for their liberality in this matter, and to the officers and workmen of the Press for the trouble they have taken.

I am indebted to Mr. W. J. Dobbs, B.A., late Scholar of St. John's College, for his kindness in reading and correcting the proof-sheets and for many valuable suggestions.

For any corrections and suggestions for imporovement I shall be thankful.

S. L. LONEY.

ROYAL HOLLOWAY COLLEGE,
EGHAM, SURREY.
September 12, 1893.

PREFACE TO THE SECOND EDITION

The Second Edition has been carefully revised, and it is hoped that few serious mistakes remain either in the text or the answers.

Some changes have been made in the chapters on logarithms and logarithmic tables, and an additional chapter has been added on Projections.

April 25, 1895.

PREFACE TO THE FIFTH EDITION

This Edition (the Seventeenth Impression) has been reset; some additions have been made to the text, and a new Table of the Radian Measure of angles has been added.

July 7, 1925

PREFACE TO THE SIXTH EDITION

Worked examples and exercises have been revised to conform to the Metric System; and with the answers is a note on their acceptable accuracy.

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THE PRINCIPAL FORMULAE IN TRIGONOMETRY

PART - I

I. Circumference of a circle $= 2\pi r$. (Art. 12.)

$\pi = 3 \cdot 14159....$ $\left[\text{Approximations are } \frac{22}{7} \text{ and } \frac{355}{113}\right]$ (Art. 13.)

A Radian $= 57°17'44 \cdot 8°$ nearly. (Art. 16.)

Two right angles $= 180° = 200^g = \pi$ radians. (Art. 19.)

$\text{Angle} = \frac{\text{arc}}{\text{radius}} \times \text{Radian}.$ (Art. 21.)

II. $\sin^2\theta + \cos^2\theta = 1;$

$\sec^2\theta = 1 + \tan^2\theta;$

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$ (Art. 27.)

III. $\sin 0° = 0; \cos 0° = 1.$ (Art. 36.)

$\sin 30° = \frac{1}{2}; \cos 30° = \frac{\sqrt{3}}{2}.$ (Art. 34.)

$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}.$ (Art. 33.)

$\sin 60° = \frac{\sqrt{3}}{2}; \cos 60° = \frac{1}{2}.$ (Art. 35.)

$\sin 90° = 1; \cos 90° = 0$ (Art. 37.)

$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}; \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$ (Art. 106.)

$\sin 18° = \frac{\sqrt{5}-1}{4}; \cos 36° = \frac{\sqrt{5}+1}{4}$ (Arts. 120, 121.)

IV. $\sin(-\theta) = -\sin\theta; \cos(-\theta) = \cos\theta.$ (Art. 68.)

$\sin(90° - \theta) = \cos\theta; \cos(90° - \theta) = \sin\theta.$ (Art. 69.)

$\sin(90° + \theta) = \cos\theta; \cos(90° + \theta) = -\sin\theta.$ (Art. 70.)

$\sin(180° - \theta) = \sin\theta; \cos(180° - \theta) = -\cos\theta.$ (Art. 72.)

$\sin(180° + \theta) = -\sin\theta; \cos(180 + \theta) = -\cos\theta.$ (Art. 73.)

V. If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ (Art. 82.)

If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha.$ (Art. 83.)

If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha.$ (Art. 84.)

VI. $\sin(A + B) = \sin A \cos B + \cos A \sin B.$

$\cos(A + B) = \cos A \cos B + \sin A \sin B.$ (Art. 88.)

$\sin(A - B) = \sin A \cos B - \cos A \sin B.$

$\cos(A - B) = \cos A \cos B + \sin A \sin B.$ (Art. 90.)

$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}.$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}.$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}.$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}. \qquad \text{(Art. 94.)}$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B).$$
$$2\cos A\cos B = \sin(A+B) - \sin(A-B).$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B).$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B). \qquad \text{(Art. 97.)}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}. \qquad \text{(Art. 98.)}$$

$$\sin 2A = 2\sin A\cos A.$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1. \qquad \text{(Art. 105.)}$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A};\ \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A} \qquad \text{(Art. 109.)}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}. \qquad \text{(Art. 105.)}$$

$$\sin 3A = 3\sin A - 4\sin^3 A.$$
$$\cos 3A = 4\cos^2 A - 3\cos A.$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}. \qquad \text{(Art. 107.)}$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}};\ \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}. \qquad \text{(Art. 110.)}$$

$$2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}.$$

$$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A}. \qquad \text{(Art. 113.)}$$

$$\tan(A_1 + A_2 + \ldots + A_n) = \frac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - \ldots}. \qquad \text{(Art. 125.)}$$

VII.

$$\log_a(mn) = \log_a m + \log_a n.$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

$$\log_a m^n = n\log_a m. \qquad \text{(Art. 136.)}$$

$$\log_a m = \log_b m \times \log_a b. \qquad \text{(Art. 147.)}$$

VIII. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$. (Art. 163.)

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, ... (Art. 164.)

$\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, ... (Art. 165.)

$\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$, ... (Art. 166.)

$\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, ... (Art. 167.)

$\sin \dfrac{A}{2} = \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$, ... (Art. 169.)

$a = b\cos C + c\cos B$, ... (Art. 170.)

$\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot \dfrac{A}{2}$, ... (Art. 171.)

$S = \sqrt{s(s-a)(s-b)(s-c)} = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ca\sin B = \dfrac{1}{2}ab\sin C$. (Art. 198.)

IX. $R = \dfrac{a}{2\sin A} = \dfrac{a}{2\sin B} = \dfrac{c}{2\sin C} = \dfrac{abc}{4S}$ (Arts. 200, 201.)

$r = \dfrac{S}{s} = (s-a)\tan \dfrac{A}{2} = \ldots = \ldots$ (Arts. 202, 203.)

$r_1 = \dfrac{S}{s-a} = s\tan \dfrac{A}{2}$. (Arts. 205, 206.)

Area of a quadrilateral inscribable in a circle
$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$. (Art. 219.)

X. $\dfrac{\sin\theta}{\theta} = 1$, when θ is very small. (Art. 228.)

Area of a circle of radius $r = \pi r^2$. (Art. 233.)

XI. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots$ to n terms

$$= \frac{\sin\left\{\alpha + \dfrac{n-1}{2}\beta\right\}\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}.$$ (Art. 241.)

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots$ to n terms

$$= \frac{\cos\left\{\alpha + \dfrac{n-1}{2}\beta\right\}\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}.$$ (Art. 242.)

CHAPTER I

MEASUREMENT OF ANGLES; SEXAGESIMAL, CENTESIMAL AND CIRCULAR MEASURE

1. In Geometry angles are measured in terms of a right angle. This, however, is an inconvenient unit of measurement on account of its size.

2. In the **Sexagesimal** system of measurement a right angle is divided into 90 equal parts called **Degrees.** Each degree is divided into 60 equal parts called **Minutes,** and each minute into 60 equal parts called **Seconds.**

The symbols 1°, 1′, and 1″ are used to denote a Degree, a Minute, and a Second respectively.

Thus 60 Seconds (60″) make One Minute (1′),
60 Minutes (60′) „ „ Degree (1°),
and 90 Degrees (90°) „ „ Right Angle.

This system is well established and is always used in the practical applications of Trigonometry. It is not however very convenient on account of the multipliers 60 and 90.

3. On this account another system of measurement called the **Centesimal,** or French, system has been proposed. In this system the right angle is divided into 100 equal parts, called **Grades** ; each grade is subdivided into 100 **Minutes**, and each minute into 100 **Seconds.**

The symbols 1^g, 1′, and 1″ are used to denote a Grade, a Minute, and a Second respectively.

Thus 100 Seconds (100″) make One Minute (1′),
100 Minutes (100′) „ „ Grade (1^g)
100 Grades (100^g) „ „ Right Angle.

4. This system would be much more convenient to use than the ordinary Sexagesimal system.

As a preliminary, however, to its practical adoption, a large number of tables would have to be recalculated. For this reason the system has in practice never been used.

5. *To convert Sexagesimal into Centesimal Measure, and vice versa.*

Since a right angle is equal to 90° and also to 100^g, we have

$$90° = 100^g$$

$$\therefore \qquad 1^\circ = \frac{10^g}{9}, \text{ and } 1^g = \frac{9^\circ}{10}.$$

Hence, to change degrees into grades, add on one-ninth ; to change grades into degrees, subtract one-tenth.

Ex. $$36^\circ = \left(36 + \frac{1}{9} \times 36\right)^g = 40^g,$$

and $$64^g = \left(64 - \frac{1}{10} \times 64\right)^\circ = (64 - 6{\cdot}4)^\circ = 57{\cdot}6^\circ.$$

If the angle does not contain an integral number of degrees, we may reduce it to a fraction of a degree and then change to grades.

In practice it is generally found more convenient to reduce any angle to a fraction of a right angle. The method will be seen in the following examples :

Ex. 1. *Reduce 63° 14′ 51″ to Centesimal Measure.*

We have $$51'' = \frac{17'}{20} = {\cdot}85',$$

and $$14'\,51'' = 14{\cdot}85' = \frac{14{\cdot}85^\circ}{60} = {\cdot}2475^\circ,$$

$$\therefore \qquad 63^\circ\,14'\,51'' = 63{\cdot}2475^\circ = \frac{63{\cdot}2475}{90} \text{ rt. angle}$$

$$= {\cdot}70275 \text{ rt. angle}$$

$$= 70{\cdot}275^g = 70^g\,27{\cdot}5' = 70^g\,27'\,50''.$$

Ex. 2. *Reduce $94^g\,23'\,97''$ to Sexagesimal Measure.*

$$94^g\,23'\,87'' = {\cdot}942387 \text{ right angle}$$

$$\frac{90}{84{\cdot}81483} \text{ degrees}$$

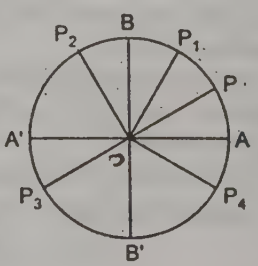
$$\frac{60}{48{\cdot}8898} \text{ minutes}$$

$$\frac{60}{53{\cdot}3880} \text{ seconds.}$$

$$\therefore \qquad 94^g\,23'\,87'' = 84^\circ\,48'\,53{\cdot}388''$$

6. Angles of any size.

Suppose AOA' and BOB' to be two fixed lines meeting at right angles in O, and suppose a revolving line OP (turning about a fixed point at O) to start from OA and revolve in a direction opposite to that of the hands of a watch.



For any position of the revolving line between OA and OB, such as OP_1, it will have turned through an angle AOP_1, which is less than a right angle.

For any position between OB and OA', such as OP_2, the angle AOP_2 through which it has turned is greater than a right angle.

For any position OP_3, between OA' and OB', the angle traced out is AOP_3, *i.e.* $AOB + BOA' + A'OP_3$, *i.e.* 2 right angles + $A'OP_3$, so that the angle described is greater than two right angles.

For any position OP_4, between OB' and OA, the angle turned through is similarly greater than three right angles.

When the revolving line has made a complete revolution, so that it coincides once more with OA, the angle through which it has turned is 4 right angles.

If the line OP still continue to revolve, the angle through which it has turned, when it is for the second time in the position OP_1, is not AOP_1 but 4 right angles + AOP_1.

Similarly, when the revolving line, having made two complete revolutions, is once more in the position OP_2, the angle it has traced out is 8 right angles + AOP_2.

7. If the revolving line OP be between OA and OB, it is said to be in the first quadrant; if it be between OB and OA', it is in the second quadrant; if between OA' and OB', it is in the third quadrant ; if it is between OB' and OA, it is in the fourth quadrant.

8. Ex. *What is the position of the revolving line when it has turned through (1) 225°, (2) 480°, and (3) 1050° ?*

(1) Since $225° = 180° + 45°$, the revolving line has turned through 45° more than two right angles, and it is therefore in the third quadrant and halfway between OA' and OB'.

(2) Since $480° = 360° + 120°$, the revolving line has turned through 120° more than one complete revolution, and is therefore in the second quadrant, *i.e.* between OB and OA', and makes an angle of 30° with OB.

(3) Since $1050° = 11 \times 90° + 60°$, the revolving line has turned through 60° more than eleven right angles, and is therefore in the fourth quadrant, *i.e.* between OB' and OA, and makes 60° with OB'.

EXAMPLES I

Express in terms of a right angle the angles :

1. 60°. **2.** 75° 15′. **3.** 63° 17′ 25″.

4. 130° 30′. **5.** 210° 30′ 30″. **6.** 370° 20′ 48″

Express in grades, minutes and seconds the angles :

7. 30°. **8.** 81°. **9.** 138° 30′.

10. 35° 47′ 15″. **11.** 235° 12′ 36″. **12.** 475° 13′ 48″.

Express in terms of right angles, and also in degrees, minutes and seconds the angles :

13. 120^g. **14.** 45^g 35′ 24″. **15.** 39^g 45′ 36″.

16. 255^g 8′ 9″ **17.** 759^g 0′ 5″.

Mark the position of the revolving line when it has traced out the following angles :

18. $\frac{4}{3}$ right angles. 19. $3\frac{1}{2}$ right angles. 20. $13\frac{1}{3}$ right angles.

21. 120°. 22. 315°. 23. 745°.

24. 1185°. 25. 150^g. 26. 420^g. 27. 875^g.

28. How many degrees, minutes and seconds are respectively passed over in $11\frac{1}{9}$ minutes by the hour and minute hands of a watch ?

29. The number of degrees in one acute angle of a right-angled triangle is equal to the number of grades in the other; express both the angles in degrees.

30. Prove that the number of Sexagesimal minutes in any angle is to the number of Centesimal minutes in the same angle as 27 : 50.

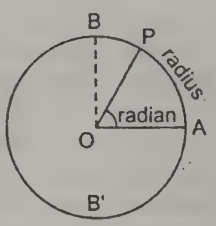
31. Divide 44° 8′ into two parts such that the number of Sexagesimal seconds in one part may be equal to the number of Centesimal seconds in the other part.

Circular Measure

9. A third system of measurement of angles has been devised, and it is this system which is used in all the higher branches of Mathematics.

The unit used is obtained thus ;

Take *any* circle *APBB'*, whose centre is *O*, and from any point *A* measure off an arc *AP* whose length is equal to the radius of the circle. Join *OA* and *OP*.



The angle *AOP* is the angle which is taken as the unit of circular measurement, *i.e.* it is the angle in terms of which in this system we measure all others.

This angle is called **A Radian** and is often denoted by 1^c.

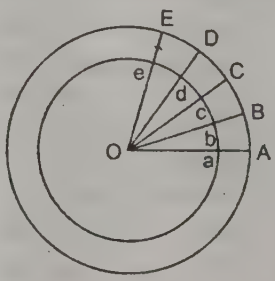
10. It is clearly essential to the proper choice of a unit that it should be a *constant* quantity ; hence we must show that the Radian is a constant angle. This we shall do in the following articles.

11. Theorem. *The length of the circumference of a circle always bears a constant ratio to its diameter.*

Take any two circles whose common centre is *O*. In the larger circle inscribe a regular polygon of *n* sides, *ABCD*.....

Let *OA* , *OB* , *OC*, ... meet the smaller circle in the points *a*, *b*, *c*, *d*, ... and join *ab*, *bc*, *cd*, ...

Then, by Geometry, *abcd*... is a regular polygon of *n* sides inscribed in the smaller circle.



Since $Oa = Ob$, and $OA = OB$, the lines ab and AB must be parallel, and hence

$$\frac{AB}{ab} = \frac{OA}{Oa}.$$

Also the outer polygon $ABCD...$ being regular, its perimeter, *i.e.* the sum of its sides, is equal to $n \,.\, AB$. Similarly for the inner polygon.

Hence we have

$$\frac{\text{Perimeter of the outer polygon}}{\text{Perimeter of the inner polygon}} = \frac{n \,.\, AB}{n \,.\, ab} = \frac{AB}{ab} = \frac{OA}{Oa} \qquad ...(1)$$

This relation exists whatever be the number of sides in the polygons.

Let then the number of sides be indefinitely increased (*i.e.* let n become inconceivably great) so that finally the perimeter of the outer polygon will be the same as the circumference of the outer circle, and the perimeter of the inner polygon the same as the circumference of the inner circle.

The relation (1) will then become

$$\frac{\text{Circumference of outer circle}}{\text{Circumference of inner circle}} = \frac{OA}{Oa}$$

$$= \frac{\text{Radius of outer circle}}{\text{Radius of inner circle}}.$$

Hence $$\frac{\text{Circumference of outer circle}}{\text{Radius of outer circle}}$$

$$= \frac{\text{Circumference of inner circle}}{\text{Radius of inner circle}}.$$

Since there was no restriction whatever as to the sizes of the two circles, it follows that the quantity

$$\frac{\text{Circumference of any circle}}{\text{Radius of that circle}}$$

is the same for all circles.

Hence the ratio of the circumference of a circle to its radius, and therefore also to its diameter, is a constant quantity.

12. In the previous article we have shown that the ratio $\frac{\text{Circumference}}{\text{Diameter}}$ is the same for all circles. The value of this constant ratio is always denoted by the Greek letter π [pronounced "Pi"], so that π is a number.

Hence $\frac{\text{Circumference}}{\text{Diameter}} =$ the constant number π

We have therefore the following theorem; **The circumference of a circle is always equal to π times its diameter or 2π times its radius.**

13. Unfortunately the value of π is not a whole number, nor can it be expressed in the form of a vulgar fraction, and hence not in the form of a decimal fraction, terminating or recurring.

The number π is an incommensurable magnitude, *i.e.* a magnitude whose value cannot be exactly expressed as the ratio of two whole numbers.

Its value, correct to 8 places of decimals, is

$$3{\cdot}14159265....$$

The fraction $\frac{22}{7}$ gives the value of π correctly for the first two decimal places; for $\frac{22}{7} = 3{\cdot}14285....$

The fraction $\frac{355}{113}$ is a more accurate value of π, being correct to 6 places of decimals; for $\frac{355}{113} = 3{\cdot}14159203...$

[N.B. The fraction $\frac{355}{113}$ may be remembered thus; write down the first three odd numbers repeating each twice, thus 113355; divide the number thus obtained into two parts and let the first part be divided into the second, thus 113)355(.

The quotient is the value of π to 6 places of decimals.]

To sum up. **An approximate value of π, correct to 2 places of decimals, is the fraction $\frac{22}{7}$; a more accurate value is $3{\cdot}14159....$**

By division, we can shew that

$$\frac{1}{\pi} = {\cdot}3183098862.....$$

14. Ex. 1. *The diameter of a wheel is 28 cm ; through what distance does its centre move during one revolution of the wheel along the ground ?*

The radius r is here 14 cm.

The circumference therefore $= 2 \,.\, \pi \,.\, 14 = 28\pi$ cm.

If we take $\pi = \frac{22}{7}$, the circumference $= 28 \times \frac{22}{7}$ cm

$= 88$ cm. approximately.

If we give π the more accurate value $3{\cdot}14159265....$, the circumference

$= 28 \times 3{\cdot}14159265...$ cm $= 87{\cdot}96459$ cm.

Ex. 2. *What must be the radius of a circular running path, round which an athlete must run 5 times in order to describe 1760 metres ?*

The circumference must be $\frac{1}{5} \times 1760$, *i.e.*, 352 metres

Hence, if r be the radius of the path in yards, we have $2\pi r = 352$,

$$r = \frac{176}{\pi} \text{ metres.}$$

Taking $\pi = \frac{22}{7}$, we have $r = \frac{176 \times 7}{22} = 56$ metres.

Taking the more accurate value $\frac{1}{\pi} = {\cdot}31831$, we have

$$r = 176 \times {\cdot}31831 = 56{\cdot}02256 \text{ metres.}$$

EXAMPLES II.

1. If the radius of the earth be 6400 km., what is the length of its circumference ?
2. The wheel of a railway carriage is 90 cm in diameter and makes 3 revolutions in a second; how fast is the train going ?
3. A mill sail whose length is 540 cm. makes 10 revolutions per minute. What distance does its end travel in an hour ?
4. The diameter of a halfpenny is an inch; what is the length of a piece of string which would just surround its curved edge ?
5. Assuming that the earth describes in one year a circle, of 1,49,700 km. radius, whose centre is the sun, how many miles does the earth travel in a year ?
6. The radius of a carriage wheel is 50 cm., and in $\frac{1}{9}$th of a second it turns through 80° about its centre, which is fixed; how many km. does a point on the rim of the wheel travel in one hour ?

15. Theorem. *The radian is a constant angle.*

Take the figure of Art. 9. Let the arc AB be a quadrant of the circle, *i.e.* one-quarter of the circumference.

By Art. 12, the length of AB is therefore $\frac{\pi r}{2}$, where r is the radius of the circle.

By Geometry, we know that angles at the centre of any circle are to one another as the arcs on which they stand.

Hence
$$\frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB} = \frac{r}{\frac{\pi r}{2}} = \frac{2}{\pi},$$

i.e.
$$\angle AOP = \frac{2}{\pi} \cdot \angle AOB.$$

But we defined the angle AOP to be a Radian.

Hence a Radian $= \frac{2}{\pi} \cdot \angle AOB$

$= \frac{2}{\pi}$ of a right angle.

Since a right angle is a constant angle, and since we have shown (Art. 12) that π is a constant quantity, it follows that a Radian is a constant angle and is therefore the same whatever be the circle from which it is derived.

16. Magnitude of a Radian.

By the previous article, a Radian

$$= \frac{2}{\pi} \times a \text{ right angle} = \frac{180°}{\pi}$$
$$= 180° \times \cdot 3183098862\ldots = 57 \cdot 2957795°$$
$$= 57° 17' 44 \cdot 8'' \text{ nearly.}$$

17. Since a Radian $= \frac{2}{\pi}$ of a right angle,

therefore a right angle $= \frac{\pi}{2}$ radians,

so that $180° = 2$ right angles $= \pi$ radians,

and $360° = 4$ right angles $= 2\pi$ radians.

Hence, when the revolving line (Art. 6) has made a complete revolution, it has described an angle equal to 2π radians; when it has made three complete revolutions, it has described an angle of 6π radians ; when it has made n revolutions, it has described an angle of $2n\pi$ radians.

18. In practice the symbol "c" is generally omitted, and instead of "an angle π^c" we find written "an angle π."

The student must notice this point carefully. If the unit, in terms of which the angle is measured, be not mentioned, he must mentally supply the word "radians." Otherwise he will easily fall into the mistake of supposing that π stands for 180°. It is true that π radians (π^c) is the same as 180°, but π itself is a number, and a number only.

19. *To convert Circular Measure into Sexagesimal Measure or Centesimal Measure and vice versa.*

The student should remember the relations

Two right angles $= 180° = 200^g = \pi$ radians.

The conversion is then merely Arithmetic.

Ex. (1) $\cdot 45\pi^c = \cdot 45 \times 180° = 81° = 90^g$.

(2) $3^c = \frac{3}{\pi} \times \pi^c = \frac{3}{\pi} \times 180° = \frac{3}{\pi} \times 200^g$.

(3) $40°\ 15'\ 36'' = 40°\ 15\frac{3}{5}' = 40 \cdot 26°$

$= 40 \cdot 26 \times \frac{\pi^c}{180} = \cdot 2236\,\pi$ radians.

(4) $40^g\ 15'\ 36'' = 40 \cdot 1536^g = 40 \cdot 1536 \times \frac{\pi}{200}$ radians

$= \cdot 200768\,\pi$ radians.

20. Ex. 1. *The angles of a triangle are in A.P. and the number of grades in the least is to the number of radians in the greatest as* $40 : \pi$ *; find the angle in degrees.*

Let the angles be $(x - y)°$, $x°$, and $(x + y)°$.

Since the sum of the three angles of a triangle is 180°, we have

$$180 = x - y + x + x + y = 3x,$$

so that $x = 60$.

The required angles are therefore

$$(60 - y)°,\ 60°,\ \text{and}\ (60 + y)°.$$

Now $(60 - y)° = \frac{10}{9} \times (60 - y)^g$,

and $(60 + y)^\circ = \frac{\pi}{180} \times (60 + y)$ radians.

Hence $\frac{10}{9}(60 - y) : \frac{\pi}{180}(60 + y) :: 40 : \pi$,

$\therefore$ $\frac{200}{\pi}\frac{(60 - y)}{60 + y} = \frac{40}{\pi}$,

i.e. $5(60 - y) = 60 + y$,

i.e. $y = 40$.

The angles are therefore 20°, 60°, and 100°.

Ex. 2. *Express in the three systems of angular measurement the magnitude of the angle of a regular decagon.*

By Geometry, we know that all the interior angles of any rectilinear figure together with four right angles are equal to twice as many right angles as the figure has sides.

Let the angle of a regular decagon contain x right angles, so that all the angles are together equal to $10x$ right angles.

The corollary therefore states that

$$10x + 4 = 20,$$

so that $x = \frac{8}{5}$ right angles.

But one right angle $= 90^\circ = 100^g = \frac{\pi}{2}$ radians.

Hence the required angle

$$= 144^\circ = 160^g = \frac{4\pi}{5} \text{ radians.}$$

EXAMPLES. III.

Express in degrees, minutes and seconds the angles :

1. $\frac{\pi^c}{3}$. **2.** $\frac{4\pi^c}{3}$. **3.** $10\pi^c$. **4.** 1^c. **5.** 8^c.

Express in grades, minutes and seconds the angles :

6. $\frac{4\pi^c}{5}$. **7.** $\frac{7\pi^c}{6}$. **8.** $10\pi^c$.

Express in radians the following angles :

9. 60°. **10.** 110° 30′. **11.** 175° 45′. **12.** 47° 25 ′ 36″.

13. 395°. **14.** 60^g. **15.** $110^g\,30'$. **16.** $345^g\,25'\,36''$.

17. The difference between the two acute angles of a right-angled triangle is $\frac{2}{5}\pi$ radians; express the angles in degrees.

18. One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees, whilst the third is $\frac{\pi x}{75}$ radians ; express them all in degrees.

19. The circular measure of two angles of a triangle are respectively $\frac{1}{2}$ and $\frac{1}{3}$; what is the number of degrees in the third angle ?

20. The angles of a triangle are in A.P. and the number of degrees in the least is to the number of radians in the greatest as 60 to π; find the angles in degrees.

21. The angles of a triangle are in A.P. and the number of radians in the least angle is to the number of degrees in the mean angle as 1 : 120. Find the angles in radians.

22. Find the magnitude, in radians and degrees, of the interior angle of (1) a regular pentagon, (2) a regular heptagon, (3) a regular octagon, (4) a regular duodecagon, and (5) a regular polygon of 17 sides.

23. The angle in one regular polygon is to that in another as 3 : 2; also the number of sides in the first is twice that in the second; how many sides have the polygons ?

24. The number of sides in two regular polygons are as 5 : 4, and the difference between their angles is 9°; find the number of sides in the polygons.

25. Find two regular polygons such that the number of their sides may be as 3 to 4 and the number of degrees in an angle of the first to the number of grades in an angle of the second as 4 to 5.

26. The angles of a quadrilateral are in A.P. and the greatest is double the least; express the least angle in radians.

27. Find in radians, degrees, and grades the angle between the hour-hand and the minute-hand of a clock at (1) half-past three, (2) twenty minutes to six, (3) a quarter past eleven.

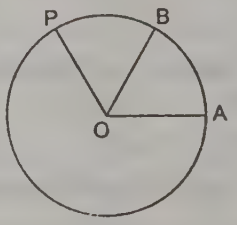
28. Find the times (1) between four and five o'clock when the angle between the minute-hand and the hour-hand is 78°, (2) between seven and eight o'clock when this angle is 54°.

21. Theorem. *The number of radians in any angle whatever is equal to a fraction, whose numerator is the arc which the angle subtends at the centre of any circle, and whose denominator is the radius of that circle.*

Let AOP be the angle which has been described by a line starting from OA and revolving into the position OP.

With centre O and any radius, describe a circle cutting OA and OP in the points A and P.

Let the angle AOB be a radian, so that the arc AB is equal to the radius OA.



By Geometry, we have

$$\frac{\angle AOP}{\text{A Radian}} = \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB} = \frac{\text{arc } AP}{\text{Radius}},$$

so that $\angle AOP = \dfrac{\text{arc } AP}{\text{Radius}}$ of a Radian.

Hence the theorem is proved.

22. Ex. 1. *Find the angle subtended at the centre of a circle of radius* 3 *cm by an arc of length* 1 *cm.*

The number of radians in the angle $= \dfrac{\text{arc}}{\text{radius}} = \dfrac{1}{3}$.

Hence the angle

$$= \frac{1}{3} \text{ radian} = \frac{1}{3} \cdot \frac{2}{\pi} \text{ right angle} = \frac{2}{3\pi} \times 90^\circ = \frac{60^\circ}{\pi} = 19\frac{1}{11}^\circ,$$

on taking π equal to $\dfrac{22}{7}$.

Ex. 2. *In a circle of* 5 *cm radius, what is the length of the arc which subtends an angle of* 33° 15′ *at the centre ?*

If x cm be the required length, we have

$$\frac{x}{5} = \text{number of radians in } 33^\circ\ 15'$$

$$= \frac{33\frac{1}{4}}{180}\pi \text{ (Art. 19)}$$

$$= \frac{133}{720}\pi.$$

$$\therefore \quad x = \frac{133}{144}\pi \text{ cm} = \frac{133}{144} \times \frac{22}{7} \text{ cm nearly}$$

$$= 2\frac{65}{72} \text{ cm nearly.}$$

Ex. 3. *Assuming the average distance of the earth from the sun to be* 149,700,000 *km and the angle subtended by the sun at the eye of a person on the earth to be* 32′, *find the sun's diameter.*

Let D be the diameter of the sun in km.

The angle subtended by the sun being very small, its diameter is very approximately equal to a small arc of a circle whose centre is the eye of the observer. Also the sun subtends an angle of 32′ at the centre of this circle.

Hence, by Art. 21, we have

$$\frac{D}{149,700,000} = \text{the number of radians in } 32'$$

$$= \text{the number of radians in } \frac{8^\circ}{15}$$

$$= \frac{8}{15} \times \frac{\pi}{180} = \frac{2\pi}{675}.$$

$$\therefore \quad D = \frac{299,400,000}{675}\pi \text{ km.}$$

$$= \frac{299,400,000}{675} \times \frac{22}{7} \text{ km approximately}$$

$$= 1,390,000 \text{ km nearly.}$$

Ex. 4. *Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find what is the height of the letters that he can read at a distance* (1) *of* 12 *metres, and* (2) 1320 *metres.*

Let x be the required height in metres.

In the first case, x is very nearly equal to the arc of a circle, of radius 12 m., which subtends an angle of 5′ at its centre.

Hence $$\frac{x}{12} = \text{number of radians in } 5'$$

$$= \frac{1}{12} \times \frac{\pi}{180}.$$

$$\therefore \quad x = \frac{\pi}{180} \text{ m} = \frac{1}{180} \times \frac{22}{7} \text{ metres nearly}$$

$$= \text{about } 17 \text{ cm}$$

In the second case, the height y is given by

$$\frac{y}{1320} = \text{number of radians in } 5'$$

$$= \frac{1}{12} \times \frac{\pi}{180},$$

so that $$y = \frac{11}{18}\pi = \frac{11}{18} \times \frac{22}{7} \text{ metres}$$

$$= \text{about } 1{\cdot}9 \text{ metres}$$

EXAMPLES. IV.

$$\left[\textit{Assume } \pi = 3{\cdot}14159 \ldots \text{ and } \frac{1}{\pi} = {\cdot}31831.\right]$$

1. Find the number of degrees subtended at the centre of a circle by an arc whose length is 0·357 times the radius.
2. Express in radians and degrees the angle subtended at the centre of a circle by an arc whose length is 15 cm, the radius of the circle being 25 cm.
3. The value of the divisions on the outer rim of a graduated circle is 5′ and the distance between successive graduations is ·1 cm. Find the radius of the circle.
4. The diameter of a graduated circle is 72 cm and the graduations on its rim are 5′ apart; find the distance from one graduation to another.
5. Find the radius of a globe which is such that the distance between two places on the same meridian whose latitude differs by 1° 10′ may be 0·5 cm.
6. Taking the radius of the earth as 64000 km, find the difference in latitude of two places, one of which is 100 km north of the other.

7. Assuming the earth to be a sphere and the distance between two parallels of latitude, which subtends an angle of 1° at the earth's centre, to be $69\frac{1}{9}$ km, find the radius of the earth.
8. The radius of a certain circle is 30 cm ; find approximately the length of an arc of this circle, if the length of the chord of the arc be 30 cm also.
9. What is the ratio of the radii of two circles at the centre of which two arcs of the same length subtend angles of 60° and 75° ?
10. If an arc, of length 10 cm., on a circle of 8 cm diameter subtend at, the centre an angle of 143° 14′ 22″, find the value of π to 4 places of decimals.
11. If the circumference of a circle be divided into 5 parts which are in A.P., and if the greatest part be 6 times the least, find in radians the magnitudes of the angles that the parts subtend at the centre of the circle.
12. The perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius; express the angle of the sector in degrees, minutes, and seconds.
13. At what distance does a man, whose height is 2 m., subtend an angle of 10′ ?
14. Find the the length which at a distnce of 5280 m. will subtend an angle of 1′ at the eye.
15. Find approximately the distance at which a globe, $5\frac{1}{2}$ cm. in diameter, will subtend an angle of 6′.
16. Find approximately the distance of a tower whose height is 51 metres and which subtends at the eye an angle of $5\frac{5}{11}'$.
17. A church spire, whose height is known to be 10 metres, subtends an angle of 9′ at the eye; find approximately its distance.
18. Find approximately in minutes the inclination to the horizon of an incline which rises $1\frac{1}{6}$ metres in 210 metres.
19. The radius of the earth being taken to be 6400 km, and the distance of the moon from the earth being 60 times the radius of the earth, find approximately the radius of the moon which subtends at the earth an angle of 16′.
20. When the moon is setting at any given place, the angle that is subtended at its centre by the radius of the earth passing through the given place is 57′. If the earth's radius be 6400 km., find approximately the distance of the moon.
21. Prove that the distance of the sun is about 130 million geographical km, assuming that the angle which the earth's radius subtends at the distance of the sun is 8·76″, and that a geographical km. subtends 1′ at the earth's centre. Find also the circumference and diameter of the earth in geographical km.
22. The radius of the earth's orbit, which is about 150000000 km., subtends at the star Sirius an angle of about ·4″; find roughly the distance of Sirius.

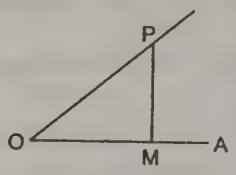
CHAPTER II

TRIGONOMETRICAL RATIOS FOR ANGLES LESS THAN A RIGHT ANGLE

23. In the present chapter we shall only consider angles which are less than a right angle.

Let a revolving line OP start from OA and revolve into the position OP, thus tracing out the angle AOP.

In the revolving line take any point P and draw PM perpendicular to the initial line OA.



In the triangle MOP, OP is the hypothenuse, PM is the perpendicular, and OM is the base.

The trigonometrical ratios, or functions, of the angle AOP are defined as follows :

$\frac{MP}{OP}$, *i.e.* $\frac{\text{Perp.}}{\text{Hyp.}}$, is called the **Sine** of the angle AOP ;

$\frac{OM}{OP}$, *i.e.* $\frac{\text{Base.}}{\text{Hyp.}}$, is called the **Cosine** of the angle AOP ;

$\frac{MP}{OM}$, *i.e.* $\frac{\text{Perp.}}{\text{Hyp.}}$, is called the **Tangent** of the angle AOP ;

$\frac{OM}{MP}$, *i.e.* $\frac{\text{Base.}}{\text{Perp.}}$, is called the **Cotangent** of the angle AOP ;

$\frac{OP}{OM}$, *i.e.* $\frac{\text{Hyp.}}{\text{Base.}}$, is called the **Secant** of the angle AOP ;

$\frac{OP}{MP}$, *i.e.* $\frac{\text{Hyp.}}{\text{Perp.}}$, is called the **Cosecant** of the angle AOP ;

The quantity by which the cosine falls short of unity, *i.e.* $1 - \cos AOP$, is called the **Versed Sine** of AOP; also the quantity $1 - \sin AOP$, by which the sine falls short of unity, is called the **Coversed Sine** of AOP

24. It will be noted that the trigonometrical ratios are all **numbers.**

The names of these eight ratios are written, for brevity, $\sin AOP$, $\cos AOP$, $\tan AOP$, $\cot AOP$, $\operatorname{cosec} AOP$, $\sec AOP$, vers AOP, and covers AOP respectively.

The two latter ratios are seldom used.

25. It will be noticed, from the definitions, that the cosecant is the reciprocal of the sine, so that

$$\operatorname{cosec} AOP = \frac{1}{\sin AOP}.$$

So the secant is the reciprocal of the cosine, *i.e.*

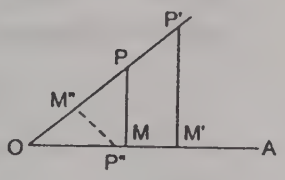
$$\sec AOP = \frac{1}{\cos AOP},$$

and the contangent is the reciprocal of the tangent, *i.e.*

$$\cot AOP = \frac{1}{\tan AOP}.$$

26. *To shew that the trigonometrical ratios are always the same for the same angle.*

We have to shew that, if in the revolving line OP any other point P' be taken and $P'M'$ be drawn perpendicular to OA, the ratios derived from the triangle $OP'M'$ are the same as those derived from the triangle OPM.



In the two triangles, the angle at O is common, and the angles at M and M' are both right angles and therefore equal.

Hence the two triangles are equiangular and therefore, by Geometry, we have $\frac{MP}{OP} = \frac{M'P'}{OP'}$, *i.e.* the sine of the angle AOP is the same whatever point we take on the revolving line.

Since, by the same proposition, we have

$$\frac{OM}{OP} = \frac{OM'}{OP'} \text{ and } \frac{MP}{OM} = \frac{M'P'}{OM'},$$

it follows that the cosine and tangent are the same whatever point be taken on the revolving line. Similarly for the other ratios.

If OA be considered as the revolving line, and in it be taken any point P'' and $P''M''$ be drawn perpendicular to OP, the functions as derived from the triangle $OP''M''$ will have the same values as before.

For, since in the two triangles OPM and $OP''M''$ the two angles $P''OM''$ and $OM''P''$ are respectively equal to POM and OMP, these two triangles are equiangular and therefore similar, and we have

$$\frac{M''P''}{OP''} = \frac{MP}{OP}, \text{ and } \frac{OM''}{OP''} = \frac{OM}{OP}.$$

27. *Fundamental relations between the trigonometrical ratios of an angle.*

We shall find that if one of the trigonometrical ratios of an angle be known, the numerical magnitude of each of the others is known also.

Let the angle AOP (fig., Art. 23) be denoted by θ [pronounced "Theta"].

In the right-angled triangle MOP we have

$$MP^2 + OM^2 = OP^2 \qquad \ldots(1)$$

Hence, dividing by OP^2, we have

$$\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1,$$

i.e. $(\sin\theta)^2 + (\cos\theta)^2 = 1.$

The quantity $(\sin\theta)^2$ is always written $\sin^2\theta$, and so for the other ratios. Hence this relation is

$$\sin^2\theta + \cos^2\theta = 1 \qquad \ldots(2)$$

Again, dividing both sides of equation (1) by OM^2, we have

$$\left(\frac{MP}{OM}\right)^2 + 1 = \left(\frac{OP}{OM}\right)^2,$$

i.e. $(\tan\theta)^2 + 1 = (\sec\theta)^2,$

so that $\sec^2\theta = 1 + \tan^2\theta$ $\qquad \ldots(3)$

Again, dividing equation (1) by MP^2, we have

$$1 + \left(\frac{OM}{MP}\right)^2 = \left(\frac{OP}{MP}\right)^2,$$

i.e. $1 + (\cot\theta)^2 = (\operatorname{cosec}\theta)^2,$

so that $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ $\qquad \ldots(4)$

Also, since $\sin\theta = \dfrac{MP}{OP}$ and $\cos\theta = \dfrac{OM}{OP}$,

we have $\dfrac{\sin\theta}{\cos\theta} = \dfrac{MP}{OP} \div \dfrac{OM}{OP} = \dfrac{MP}{OM} = \tan\theta.$

Hence $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad \ldots(5)$

Similarly $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $\qquad \ldots(6)$

28. Ex. 1. *Prove that* $\sqrt{\dfrac{1-\cos A}{1+\cos A}} = \operatorname{cosec} A - \cot A.$

We have $\sqrt{\dfrac{1-\cos A}{1+\cos A}} = \sqrt{\dfrac{(1-\cos A)^2}{1-\cos^2 A}}$

$$= \frac{1-\cos A}{\sqrt{1-\cos^2 A}} = \frac{1-\cos A}{\sin A},$$

by relation (2) of the last article,

$$= \frac{1}{\sin A} - \frac{\cos A}{\sin A} = \operatorname{cosec} A - \cot A.$$

Ex. 2. *Prove that*

$$\sqrt{\sec^2 A + \operatorname{cosec}^2 A} = \tan A + \cot A.$$

We have seen that $\sec^2 A = 1 + \tan^2 A,$

and $\operatorname{cosec}^2 A = 1 + \cot^2 A.$

$$\therefore \sec^2 A + \operatorname{cosec}^2 A = \tan^2 A + 2 + \cot^2 A$$
$$= \tan^2 A + 2\tan A \cot A + \cot^2 A$$
$$= (\tan A + \cot A)^2,$$

so that $\sqrt{\sec^2 A + \operatorname{cosec}^2 A} = \tan A + \cot A.$

Ex. 3. *Prove that*

$$(\operatorname{cosec} A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1.$$

The given expression

$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$$
$$= \frac{1 - \sin^2 A}{\sin A} \cdot \frac{1 - \cos^2 A}{\cos A} \cdot \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$$
$$= \frac{\cos^2 A}{\sin A} \cdot \frac{\sin^2 A}{\cos A} \cdot \frac{1}{\sin A \cos A}$$
$$= 1$$

EXAMPLES. V.

Prove the following statements :

1. $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A.$
2. $(\sin A + \cos A)(1 - \sin A \cos A) = \sin^3 A + \cos^3 A.$
3. $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A} = 2\operatorname{cosec} A.$
4. $\cos^6 A + \sin^6 A = 1 - 3\sin^2 A \cos^2 A.$
5. $\sqrt{\dfrac{1 - \sin A}{1 + \sin A}} = \sec A - \tan A.$
6. $\dfrac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \dfrac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} = 2\sec^2 A.$
7. $\dfrac{\operatorname{cosec} A}{\cot A + \tan A} = \cos A.$
8. $(\sec A + \cos A)(\sec A - \cos A) = \tan^2 A + \sin^2 A.$
9. $\dfrac{1}{\cot A + \tan A} = \sin A \cos A.$
10. $\dfrac{1}{\sec A - \tan A} = \sec A + \tan A.$
11. $\dfrac{1 - \tan A}{1 + \tan A} = \dfrac{\cot A - 1}{\cot A + 1}.$
12. $\dfrac{1 + \tan^2 A}{1 + \cot^2 A} = \dfrac{\sin^2 A}{\cos^2 A}.$

13. $\dfrac{\sec A - \tan A}{\sec A + \tan A} = 1 - 2\sec A \tan A + 2\tan^2 A.$

14. $\dfrac{\tan A}{1 - \cot A} + \dfrac{\cot A}{1 - \tan A} = \sin A \operatorname{cosec} A + 1.$

15. $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} = \sin A + \cos A.$

16. $(\sin A + \cos A)(\cot A + \tan A) = \sec A + \operatorname{cosec} A.$

17. $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A.$

18. $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A.$

19. $\sqrt{\operatorname{cosec}^2 A - 1} = \cos A \operatorname{cosec} A.$

20. $\sec^2 A \operatorname{cosec}^2 A = \tan^2 A + \cot^2 A + 2.$

21. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A.$

22. $(1 + \cot A - \operatorname{cosec} A)(1 + \tan A + \sec A) = 2.$

23. $\dfrac{1}{\operatorname{cosec} A - \cot A} - \dfrac{1}{\sin A} = \dfrac{1}{\sin A} = \dfrac{1}{\operatorname{cosec} A + \cot A},$

24. $\dfrac{\cot A \cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A \cos A}$

25. $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \tan B.$

26. $\left(\dfrac{1}{\sec^2 a - \cos^2 a} + \dfrac{1}{\operatorname{cosec}^2 a - \sin^2 a}\right) \cos^2 a \sin^2 a = \dfrac{1 - \cos^2 a \sin^2 a}{2 + \cos^2 a \sin^2 a}.$

27. $\sin^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A \cos^2 A).$

28. $\dfrac{\cos A \operatorname{cosec} A - \sin A \sec A}{\cos A + \sin A} = \operatorname{cosec} A - \sec A.$

29. $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}.$

30. $(\tan a + \operatorname{cosec} \beta)^2 - (\cot \beta - \sec a)^2 = 2 \tan a \cot \beta (\operatorname{cosec} a + \sec \beta)$

31. $2\sec^2 a - \sec^4 a - 2\operatorname{cosec}^2 a + \operatorname{cosec}^4 a = \cot^4 a - \tan^4 a$

32. $(\sin a + \operatorname{cosec} a)^2 + (\cos a + \sec a)^2 = \tan^2 a + \cot^2 a + 7.$

33. $(\operatorname{cosec} A + \cot A) \operatorname{covers} A - (\sec A + \tan A) \operatorname{vers} A = (\operatorname{cosec} A - \sec A)(2 - \operatorname{vers} A \operatorname{covers} A).$

34. $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\operatorname{cosec}^2 A} - \dfrac{\operatorname{cosec} A}{\sec^2 A}.$

35. $2 \operatorname{versin} A + \cos^2 A = 1 + \operatorname{versin}^2 A$

29. *Limits to the values of the trigonometrical ratios.*

From equation (2) of Art. 27, we have

$$\sin^2 \theta + \cos^2 \theta = 1.$$

Now $\sin^2\theta$ and $\cos^2\theta$, being both squares, are both necessarily positive. Hence, since their sum is unity, neither of them can be greater than unity.

[For if one of them, say $\sin^2\theta$, were greater than unity, the other, $\cos^2\theta$, would have to be negative, which is impossible.]

Hence neither the sine nor the cosine can be numerically greater than unity.

Since $\sin\theta$ cannot be greater than unity, therefore $\operatorname{cosec}\theta$, which equals $\frac{1}{\sin\theta}$, cannot be numerically less than unity.

So $\sec\theta$, which equals $\frac{1}{\cos\theta}$, cannot be numerically less than unity.

30. The foregoing results follow easily from the figure of Art. 23.

For, whatever be the value of the angle AOP, neither the side OM nor the side MP is ever greater than OP.

Since MP is never greater than OP, the ratio $\frac{MP}{OP}$ is never greater than unity, so that the sine of an angle is never greater than unity.

Also, since OM is never greater than OP, the ratio $\frac{OM}{OP}$ is never greater than unity, *i.e.* the cosine is never greater than unity.

31. We can express the trigonometrical ratios of an angle in terms of any of them.

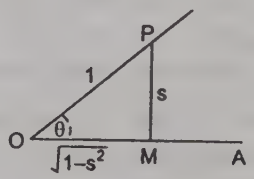
The simplest method of procedure is best shown by examples.

Ex. 1. *To express all the trigonometrical ratios in terms of the sine.*

Let AOP be any angle θ.

Let the length OP be unity and let the corresponding length of MP be s.

Then $OM = \sqrt{OP^2 - MP^2} = \sqrt{1 - s^2}$.



Hence $$\sin\theta = \frac{MP}{OP} = \frac{s}{1} = s,$$

$$\cos\theta = \frac{OM}{OP} = \sqrt{1 - s^2} = \sqrt{1 - \sin^2\theta},$$

$$\tan\theta = \frac{MP}{OM} = \frac{s}{\sqrt{1 - s^2}} = \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}},$$

$$\cot\theta = \frac{OM}{MP} = \frac{\sqrt{1 - s^2}}{s} = \frac{\sqrt{1 - \sin^2\theta}}{\sin\theta},$$

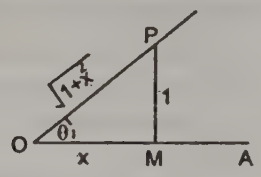
$$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{1}{s} = \frac{1}{\sin\theta}$$

and $$\sec\theta = \frac{OP}{OM} = \frac{1}{\sqrt{1 - s^2}} = \frac{1}{\sqrt{1 - \sin^2\theta}}.$$

The last five equations give what is required.

Ex. 2. *To express all the trigonometrical ratios in terms of the cotangent.*

Taking the usual figure, let the length MP be unity, and let the corresponding value of OM be x.



Then

$$OP = \sqrt{OM^2 + MP^2} = \sqrt{1 + x^2}.$$

Hence
$$\cot\theta = \frac{OM}{MP} = \frac{x}{1} = x,$$

$$\sin\theta = \frac{MP}{OP} = \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+\cot^2\theta}},$$

$$\cos\theta = \frac{OM}{OP} = \frac{x}{\sqrt{1+x^2}} = \frac{\cot\theta}{\sqrt{1+\cot^2\theta}},$$

$$\tan\theta = \frac{MP}{OM} = \frac{1}{x} = \frac{1}{\cot\theta},$$

$$\sec\theta = \frac{OP}{OM} = \frac{\sqrt{1+x^2}}{x} = \frac{\sqrt{1+\cot^2\theta}}{\cot\theta},$$

and
$$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+\cot^2\theta}.$$

The last five equations give what is required.

It will be noticed that, in each case, the denominator of the fraction which defines the trigonometrical ratio was taken equal to unity. For example, the sine is $\frac{MP}{OP}$, and hence in Ex. 1 the denominator OP is taken equal to unity.

The cotangent is $\frac{OM}{MP}$, and hence in Ex. 2. the side MP is taken equal to unity.

Similarly suppose we had to express the other ratios in terms of the cosine, we should, since the cosine is equal to $\frac{OM}{OP}$, put OP equal to unity and OM equal to c. The working would then be similar to that of Exs. 1 and 2.

In the following examples the sides have numerical values.

Ex. 3. If $\cos\theta$ equal $\frac{3}{5}$, find the values of the other ratios.

Along the initial line OA take OM equal to 3, and erect a perpendicular MP.

Let a line OP, of length 5, revolve round O until its other end meets this perpendicular in the point P. Then AOP is the angle θ.

By Geometry, $MP = \sqrt{OP^2 - OM^2} = \sqrt{5^2 - 3^2} = 4$.

Hence clearly

$$\sin\theta = \frac{4}{5}, \tan\theta = \frac{4}{3}, \cot\theta = \frac{3}{4}, \operatorname{cosec}\theta = \frac{5}{4}, \text{ and } \sec\theta = \frac{5}{3}.$$

Ex. 4. *Supposing θ to be an angle whose sine is $\frac{1}{3}$, to find the numerical magnitude of the other trigonometrical ratios.*

Here $\sin\theta = \frac{1}{3}$, so that the relation (2) of Art. 27 gives

$$\left(\frac{1}{3}\right)^2 + \cos^2\theta = 1,$$

i.e. $\cos^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$ *i.e.* $\cos\theta = \frac{2\sqrt{2}}{3}$.

Hence $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

$$\cot\theta = \frac{1}{\tan\theta} = 2\sqrt{2},$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = 3,$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4},$$

$$\operatorname{vers}\theta = 1 - \cos\theta = 1 - \frac{2\sqrt{2}}{3},$$

and $\operatorname{covers}\theta = 1 - \sin\theta = 1 - \frac{1}{3} = \frac{2}{3}$.

32. In the following table is given the result of expressing each trigonometrical ratio in terms of each of the others.

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

EXAMPLES. VI.

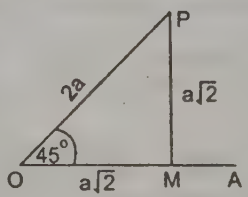
1. Express all the other trigonometrical ratios in terms of the cosine.
2. Express all the ratios in terms of the tangent.
3. Express all the ratios in terms of the cosecant.
4. Express all the ratios in terms of the secant.
5. The sine of a certain angle is $\frac{1}{4}$; find the numerical values of the other trigonometrical ratios of this angle.
6. If $\sin\theta = \frac{12}{13}$, find $\tan\theta$ and versin θ.
7. If $\sin A = \frac{11}{61}$, find tan A, cos A and sec A.
8. If $\cos\theta = \frac{4}{5}$, find $\sin\theta$ and $\cot\theta$.
9. If $\cos A = \frac{9}{41}$, find tan A and cosec A.
10. If $\tan\theta = \frac{3}{4}$, find the sine, cosine, versine, and cosecant of θ.
11. If $\tan\theta = \frac{1}{\sqrt{7}}$ find the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$.
12. If $\cot\theta = \frac{15}{8}$, find $\cos\theta$ and cosec θ.
13. If $\sec A = \frac{3}{2}$, find $\tan A$ and $\operatorname{cosec} A$.
14. If $2\,\mathrm{Sin}\,\theta = 2 - \cos\theta$, find $\sin\theta$.
15. If $8\,\mathrm{Sin}\,\theta = 4 + \cos\theta$, find $\sin\theta$.
16. If $\tan\theta + \sec\theta = 1.5$, find $\sin\theta$.
17. If $\cot\theta + \operatorname{cosec}\theta = 5$, find $\cos\theta$.
18. If $3\sec^4\theta + 8 = 10\sec^2\theta$, find the values of $\tan\theta$.
19. If $\tan^2\theta + \sec\theta = 5$, find $\cos\theta$
20. If $\tan\theta + \cot\theta = 2$, find $\sin\theta$.
21. If $\sec^2\theta = 2 + 2\tan\theta$, find $\tan\theta$.
22. If $\tan\theta = \frac{2x(x+1)}{2x+1}$ find $\sin\theta$ and $\cos\theta$.

Values of the trigonometrical ratios in some useful cases.

33. *Angle of* 45°

Let the angle AOP traced out be 45°.

Then, since the three angles of a triangle are together equal to two right angles,



$$\angle OPM = 180° - \angle POM - \angle PMO$$
$$= 180° - 45° - 90° = 45° = \angle POM.$$
$$\therefore \quad OM = MP.$$

If $OP = 2a$, we have

$$4a^2 = OP^2 = OM^2 + MP^2 = 2.OM^2,$$

so that $OM = a\sqrt{2}$.

$$\therefore \quad \sin 45° = \frac{MP}{OP} = \frac{a\sqrt{2}}{2a} = \frac{1}{\sqrt{2}}$$

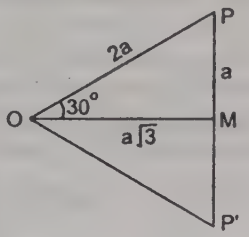
$$\cos 45° = \frac{OM}{OP} = \frac{a\sqrt{2}}{2a} = \frac{1}{\sqrt{2}},$$

and $\tan 45° = 1$.

34. *Angle of* 30°.

Let the angle AOP traced out be 30°.

Produce PM to P' making MP' equal to PM.



The two triangles OMP and OMP' have their sides OM and MP equal to OM and MP' and also the contained angles equal.

Therefore $OP' = OP$, and $\angle OP'P = \angle OPP' = 60°$, so that the triangle $P'OP$ is equilateral.

Hence, if $OP = 2a$, we have

$$MP = \tfrac{1}{2}P'P = \tfrac{1}{2}OP = a.$$

Also $OM = \sqrt{OP^2 - MP^2} = \sqrt{4a^2 - a^2} = a\sqrt{3}$.

$$\therefore \quad \sin 30° = \frac{MP}{OP} = \frac{1}{2},$$

$$\cos 30° = \frac{OM}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2},$$

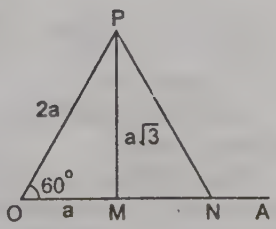
and $$\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{1}{\sqrt{3}}.$$

35. *Angle of* 60°

Let the angle AOP traced out be 60°.

Take a point N on OA, so that

$$MN = OM = a \text{ (say)}.$$



The two triangles OMP and NMP have now the sides OM and MP equal to NM and MP respectively, and the included angles equal, so that the triangles are equal.

$\therefore PN = OP$, and $\angle PNM = \angle POM = 60°$.

The triangle OPN is therefore equilateral, and hence

$$OP = ON = 2OM = 2a.$$

$$\therefore \quad MP = \sqrt{OP^2 - OM^2} = \sqrt{4a^2 - a^2} = a\sqrt{3}$$

Hence $\sin 60° = \frac{MP}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$,

$\cos 60° = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2}$,

and $\tan 60° = \frac{\sin 60°}{\cos 60°} = \sqrt{3}$.

36. *Angle of* 0°.

Let the revolving line OP have turned through a very small angle, so that the angle MOP is very small.

The magnitude of MP is then very small, and initially, before OP had turned through an angle large enough to be perceived, the quantity MP was smaller than any quantity we could assign, *i.e.* was what we denote by 0.

Also, in this case, the two points M and P very nearly coincide, and the smaller the angle AOP the more nearly do they coincide.

Hence, when the angle AOP is actually zero, the two lengths OM and OP are equal and MP is zero.

Hence $\sin 0° = \frac{MP}{OP} = \frac{0}{OP} = 0$,

$\cos 0° = \frac{OM}{OP} = \frac{OP}{OP} = 1$,

and $\tan 0° = \frac{0}{1} = 0$.

Also $\cot 0°$ = the value of $\frac{OM}{MP}$ when M and P coincide

= the ratio of a finite quantity to something infinitely small

= a quantity which is infinitely great.

Such a quantity is usually denoted by the symbol ∞.

Hence $\cot 0° = \infty$.

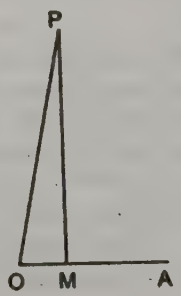
Similarly $\operatorname{cosec} 0° = \frac{OP}{MP} = \infty$ also.

And $\sec 0° = \frac{OP}{OM} = 1$.

37. *Angle of* 90°.

Let the angle AOP by very nearly, but not quite, a right angle.

When OP has actually described a right angle, the point M coincides with O, so that then OM is zero and OP and MP are equal.



Hence $\sin 90° = \frac{MP}{OP} = \frac{OP}{OP} = 1$,

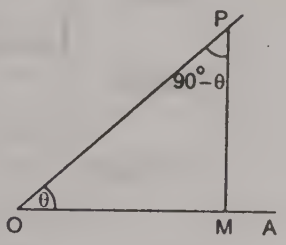
$\cos 90° = \frac{OM}{OP} = \frac{0}{OP} = 0$,

$$\tan 90° = \frac{MP}{OM} = \frac{\text{a finite quantity}}{\text{an infinitely small quantity}}$$
$$= \text{a number infinitely large} = \infty,$$
$$\cot 90° = \frac{OM}{MP} = \frac{0}{MP} = 0,$$
$$\sec 90° = \frac{OP}{OM} = \infty, \text{ as in the case of the tangent,}$$
and $$\operatorname{cosec} 90° = \frac{OP}{MP} = \frac{OP}{OP} = 1.$$

38. Complementary Angles. Def. Two anlges are said to be complementary when their sum is equal to a right angle. Thus any angle θ and the angle $90° - \theta$ are complementary.

39. *To find the relations between the trigonometrical ratios of two complementary angles.*

Let the revolving line, strating from OA, trace out any acute angle AOP, equal to θ. From any point P on it draw PM perpendicular to OA.



Since the three angles of a triangle are together equal to two right angles, and since OMP is a right angle, the sum of the two angles MOP and OPM is a right angle.

They are therefore complementary and

$$\angle OPM = 90° - \theta.$$

[When the angle OPM is considered, the line PM is the "base" and MO is the "perpendicular."]

We then have

$$\sin (90° - \theta) = \sin MPO = \frac{MO}{PO} = \cos AOP = \cos \theta,$$
$$\cos (90° - \theta) = \cos MPO = \frac{PM}{PO} = \sin AOP = \sin \theta,$$
$$\tan (90° - \theta) = \tan MPO = \frac{MO}{PM} = \cot AOP = \cot \theta,$$
$$\cot (90° - \theta) = \cot MPO = \frac{PM}{MO} = \tan AOP = \tan \theta,$$
$$\operatorname{cosec} (90° - \theta) = \operatorname{cosec} MPO = \frac{PO}{MO} = \sec AOP = \sec \theta,$$
and $$\sec (90° - \theta) = \sec MPO = \frac{PO}{PM} = \operatorname{cosec} AOP = \operatorname{cosec} \theta.$$

Hence we observe that

the Sine of any angle = the Cosine of its complement, the Tangent of any angle

= the Cotangent of its complement,

and the Secant of any angle

= the Cosecant of its complement.

From this is apparent what is the derivation of the names Cosine, Cotangent, and Cosecant.

40. The student is advised before proceeding any further to make himself quite familiar with the following table. [For an extension of this table, see Art. 76.]

Angle	0°	30°	45°	60°	90°
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
Cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
Cosecant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
Secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

If the student commits accurately to memory the portion of the above table included between the thick lines, he should be able to easily reproduce the rest.

For

(1) the sines of 60° and 90° are respectively the cosines of 30° and 0°, and (Art. 39.)

(2) The cosines of 60° and 90° are respectively the sines of 30° and 0°. (Art. 39.)

Hence the second and third lines are known.

(3) The tangent of any angle is the result of dividing the sine by the cosine.

Hence any quantity in the fourth line is obtained by dividing the corresponding quantity in the second line by the corresponding quantity in the third line.

(4) The cotangent of any angle is the reciprocal of the tangent, so that the quantities in the fifth row are the reciprocals of the quantities in the fourth row.

(5) Since $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, the sixth row is obtained by inverting the corresponding quantities in the second row.

(6) Since $\sec\theta = \frac{1}{\cos\theta}$, the seventh row is similarly obtained from the third row.

EXAMPLES. VII.

1. If $A = 30°$, verify that

(1) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$,

(2) $\sin 2A = 2\sin A \cos A$,

(3) $\cos 3A = 4\cos^3 A - 3\cos A$,

(4) $\sin 3A = 3\sin A - 4\sin^3 A$,

and (5) $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$.

2. If $A = 45°$, verify that

(1) $\sin 2A = 2\sin A \cos A$,

(2) $\cos 2A = 1 - 2\sin^2 A$,

and (3) $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$.

Verify that

3. $\sin^2 30° + \sin^2 45° + \sin^2 60° = \frac{3}{2}$.

4. $\tan^2 30° + \tan^2 45° + \tan^2 60° = 4\frac{1}{3}$.

5. $\sin 30° \cos 60° + \cos 30° \sin 60° = 1$.

6. $\cos 45° \cos 60° - \sin 45° \sin 60° = -\frac{\sqrt{3} - 1}{2\sqrt{2}}$.

7. $\frac{4}{3}\cot^2 30° + 3\sin^2 60° - 2\operatorname{cosec}^2 60° - \frac{3}{4}\tan^2 30° = 3\frac{1}{3}$.

8. $\operatorname{cosec}^2 45° \cdot \sec^2 30°, \sin^2 90°, \cos 60° = 1\frac{1}{3}$.

9. $4\cot^2 45° - \sec^2 60° + \sin^3 30° = \frac{1}{8}$.

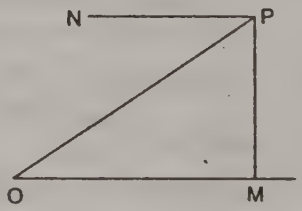
CHAPTER III

SIMPLE PROBLEMS IN HEIGHTS AND DISTANCES

41. One of the objects of Trigonometry is to find the distances between points, or the heights of objects, without actually measuring these distances or these heights.

42. Suppose O and P to be two points, P being at a higher level than O.

Let OM be a horizontal line drawn through O to meet in M the vertical line drawn through P.



The angle MOP is called the **Angle of Elevation** of the point P as seen from O.

Draw PN parallel to MO, so that PN is the horizontal line passing through P. The angle NPO is the **Angle of Depression** of the point O as seen from P.

43. Two of the instruments used in practical work are the Theodolite and the Sextant.

The Theodolite is used to measure angles in a vertical plane.

In its simple form it consists of a telescope attached to a flat piece of wood. This piece of wood is supported by three legs and can be arranged so as to be accurately horizontal.

This table being at O and horizontal, and the telescope being initially pointing in the direction OM, the latter can be made to rotate in a vertical plane until it points accurately towards P. A graduated scale shews the angle through which it has been turned from the horizontal, *i.e.* gives us the angle of elevation MOP.

Similarly, if the instrument were at P, the angle NPO through which the telescope would have to be turned, downward from the horizontal, would give us the angle NPO.

The instrument can also be used to measure angles in a horizontal plane.

44. The Sextant is used to find the angle subtended by any two points D and E at a third point F. It is an instrument much used on board ships.

Its construction and application are too complicated to be here considered.

45. We shall now solve a few simple examples in heights and distances.

Ex. 1. *A vertical flagstaff stands on a horizontal plane ; from a point distant 60 metres from its foot, the angle of elevation of its top is found to be 30°; find the height of the flagstaff.*

Let MP (fig., Art. 42) represent the flagstaff and O the point from which the angle of elevation is taken.

Then $OM = 60$ metres, and $\angle MOP = 30°$.

Since PMO is a right angle, we have

$$\frac{MP}{OM} = \tan MOP = \tan 30° = \frac{1}{\sqrt{3}} \quad \text{(Art. 34)}.$$

$$\therefore \quad MP = \frac{OM}{\sqrt{3}} = \frac{60}{\sqrt{3}} = \frac{60\sqrt{3}}{3} = 20\sqrt{3}.$$

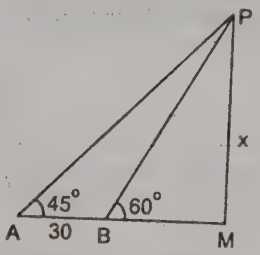
Now, by extraction of the square root, we have

$$\sqrt{3} = 1{\cdot}73205\ldots .$$

Hence $MP = 20 \times 1{\cdot}73205\ldots$ metres $= 34{\cdot}641\ldots$ metres.

Ex. 2. *A man wishes to find the height of a church spire which stands on a horizontal plane ; at a point on this plane he finds the angle of elevation of the top of the spire to be 45° ; on walking 30 metres toward the lower he finds the corresponding angle of elevation to be 60° ; deduce the height of the tower and also his original distance from the foot of the spire.*

Let P be the top of the spire and A and B the two points at which the angles of elevation are taken. Draw PM perpendicular to AB prodneed and let MP be x.



We are given $AB = 30$ metres,

$$\angle MAP = 45°,$$

and $$\angle MBP = 60°.$$

We then have

$$\frac{AM}{x} = \cot 45° = 1,$$

and

$$\frac{BM}{x} = \cot 60° = \frac{1}{\sqrt{3}}$$

Hence $AM = x$, and $BM = \frac{x}{\sqrt{3}}$,

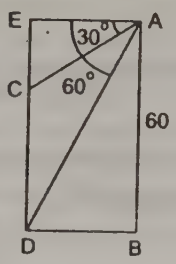
$$\therefore \quad 30 = AM - BM = x - \frac{x}{\sqrt{3}} = x\frac{\sqrt{3}-1}{\sqrt{3}}.$$

$$\therefore \quad x = \frac{30\sqrt{3}}{\sqrt{3}-1} = \frac{30\sqrt{3}(\sqrt{3}+1)}{3-1} = 30(3+\sqrt{3})$$

$$= 15[3 + 1{\cdot}73205\ldots] = 71\ldots \text{ metres}$$

Also $AM = x$, so that both of the required distances are equal to 71 metres.

Ex. 3. *From the top of a cliff, 60 metres high, the angles of depression of the top and bottom of a tower are observed to be 30° and 60°; find the height of the tower.*



Let A be the point of observation and BA the height of the cliff, and let CD be the tower.

Draw AE horizontally, so that $\angle EAC = 30°$ and $\angle EAD = 60°$.

Let x metres be the height of the tower and produce DC to meet AE in E, so that $CE = AB - x = 60 - x$.

Since $\angle ADB = \angle DAE = 60°$,

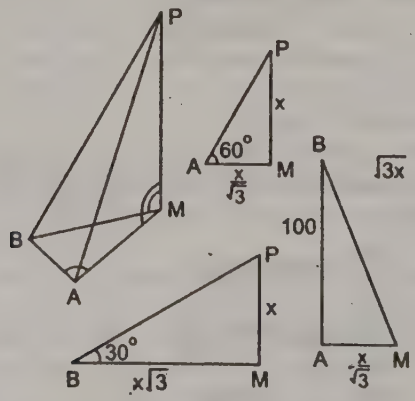
$$\therefore DB = AB \cot ADB = 60 \cot 60° = \frac{60}{\sqrt{3}}$$

Also $$\frac{60 - x}{DB} = \frac{CE}{EA} = \tan 30° = \frac{1}{\sqrt{3}}.$$

$$\therefore \quad 60 - x = \frac{DB}{\sqrt{3}} = \frac{60}{3},$$

so that $$x = 60 - \frac{60}{3} = 40 \text{ metres.}$$

Ex. 4. *A man observes that at a point due south of a certain tower its angle of elevation is 60° ; he then walks 100 metres due west on a horizontal plane and finds that the angle of elevation is 30° ; find the height of the tower and his original distnce from it.*



Let P be the top, and PM the height of the tower, A the point due south of the tower and B the point due west of A.

The angle PMA, PMB and MAB are therefore all right angles.

For simplicity, since the triangles PAM , PBM and ABM are in different planes, they are reproduced in the second, third and fourth figures and drawn to scale.

We are given $AB = 100$ metres, $\angle PAM = 60°$, and $\angle PBM = 30°$,

Let the height of the tower be x metres.

From the second figure,

$$\frac{AM}{x} = \cot 60° = \frac{1}{\sqrt{3}},$$

so that $$AM = \frac{x}{\sqrt{3}}.$$

From the third figure,

$$\frac{BM}{x} = \cot 30° = \sqrt{3},$$

so that $$BM = x\sqrt{3}.$$

From the last figure, we have

$$BM^2 = AM^2 + AB^2,$$

i.e., $$3x^2 = \frac{1}{3}x^2 + 100^2.$$

$\therefore$ $$8x^2 = 3 \times 100^2.$$

$\therefore$ $$x = \frac{100\sqrt{3}}{2\sqrt{2}} = \frac{50\sqrt{6}}{2} = 25 \times \sqrt{6}$$

$$= 25 \times 2{\cdot}44949 \ldots = 61{\cdot}24 \text{ metres.}$$

Also his original distance from the tower

$$= x \cot 60° = \frac{x}{\sqrt{3}} = 25 \times \sqrt{2}$$

$$= 25 \times (1{\cdot}4142\ldots) = 35{\cdot}35 \ldots \text{ metres.}$$

EXAMPLES VIII.

1. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60° ; when he retires 20 metres from the bank, he finds the angle to be 30° ; find the height of the tree and the breadth of the river.
2. At a certain point the angle of elevation of a tower is found to be such that its cotangent is $\frac{3}{5}$; on walking 32 metres directly toward the tower its angle of elevation is an angle whose cotangent is $\frac{2}{5}$. Find the height of the tower.
3. At a point A, the angle of elevation of a tower is found to be such that its tangent is $\frac{5}{12}$; on walking 240 metres nearer the tower the tangent of the angle of elevation is found to be $\frac{3}{4}$; what is the height of the tower ?
4. Find the height of a chimney when it is found that, on walking towards it 50 metres in a horizontal line through its base, the angular elevation of its top changes from 30° to 45°.

5. An observer on the top of a cliff, 200 metres above the sea-level, observes the angles of depression of two ships at anchor to be 45° and 30° respectively; find the distances between the ships if the line joining them points to the base of the cliff.
6. From the top of a cliff an observer finds that the angles of depression of two buoys in the sea are 39° and 26° respectively; the buoys are 300 metres apart and the line joining them points straight at the foot of the cliff; find the height of the cliff and the distance of the nearest buoy from the foot of the cliff, given that cot 26° = 2·0503, and cot 39° = 1·2349.
7. The upper part of a tree broken over by the wind makes an angle of 30° with the ground, and the distance from the root to the point where the top of the tree touches the ground is 10 m., what was the height of the tree ?
8. The horizontal distance between two towers is 60 metres and the angular depression of the top of the first as seen from the top of the second, which is 150 m high, is 30°; find the height of the first.
9. The angle of elevation of the top of an unfinished tower at a point distant 120 metres from its base is 45°; how much higher must the tower be raised so that its angle of elevation at the same point may be 60°?
10. Two pillars of equal height stand on either side of a wide road which is 100 metres wide; at a point in the road between the towers the elevation of the tops of the towers are 60° and 30°; find their height and the position of the point.
11. The angle of elevation of the top of a tower is observed to be 60°; at a point 40 metre above the first point of observation the elevation is found to be 45°; find the height of the tower and its horizontal distance from the points of observation.
12. At the foot of a mountain the elevation of its summit is found to be 45°; after ascending 1000 m towards the mountain up a slope of 30° inclination the elevation is found to be 60°. Find the height of the mountain.
13. What is the angle of elevation of the sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole ?
14. The shadow of a tower standing on a level plane is found to be 60 metres longer when the sun's altitude is 30° than when it is 45°. Prove that the height of the tower is $30\,(1 + \sqrt{3})$ metres.
15. On a straight coast there are three objects A, B and C, such that $AB = BC = 2$ km. A vessel approaches B in a line perpendicular to the coast, and at a certain point AC is found to subtend an angle of 60°; after sailing in the same direction for ten minutes AC is found to subtend an angle of 120°; find the rate at which the ship is going.
16. Two flagstaffs stand on a horizontal plane. A and B are two points on the line joining the bases of the flagstaffs and between them. The angles of elevation of the tops of the flagstaffs as seen from A are 30° and 60° and, as seen from B, they are 60° and 45°. If the length AB be 10 metres, find the heights of the flagstaffs and the distance between them.

17. P is the top and Q the foot of a tower standing on a horizontal plane. A and B are two points on this plane such that AB is 32 m and QAB is a right angle. It is found that

$$\cot PAQ = \frac{2}{5}, \text{ and } \cot PBQ = \frac{3}{5};$$

find the height of the tower.

18. A square tower stands upon a horizontal plane. From a point in this plane, from which three of its upper corners are visible, their angular elevations are respectively 45°, 60° and 45°. Shew that the height of the tower is to the breadth of one of its sides as $\sqrt{6}\ (\sqrt{5} + 1)$ to 4.

19. A lighthouse, facing north, sends out a fan-shaped beam of light extending from north-east to north-west. An observer on a steamer, sailing due west, first sees the light when he is 5 km away from the lighthouse and continues to see it for $30\sqrt{2}$ minutes. What is the speed of the steamer ?

20. A man stands at a point X on the bank XY of a river with straight and parallel banks, and observes that the line joining X to a point Z on the opposite bank makes an angle of 30° with XY. He then goes along the bank a distance of 200 metres to Y and finds that the angle ZYX is 60°. Find the breadth of the river.

21. A man, walking due north, observes that the elevation of a balloon, which is due east of him and is sailing toward the north-west, is then 60°, after he has walked 400 metres the balloon is vertically over his head; find its height supposing it to have always remained the same.

CHAPTER IV

APPLICATION OF FALGEBRAIC SIGNS TO TRIGONOMETRY

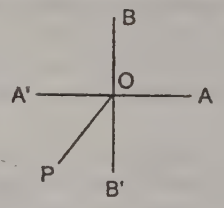
46. Positive and Negative Angles. In Art. 6, in treating of angles of any size, we spoke of the revolving line as if it always revolved in a direction opposite to that in which the hands of a watch revolve, when the watch is held with its face uppermost.

This direction is called counter-clockwise.

When the revolving line turns in this manner it is said to revolve in the positive direction and to trace out a positive angle.

When the line OP revolves in the opposite direction, *i.e.* in the same direction as the hands of the watch, it is said to revolve in the negative direction and to trace out a negative angle. This negative direction is clockwise.

47. Let the revolving line start from OA and revolve until it reaches a position OP, which lies between OA' and OB' and which bisects the angle $A'OB'$.



If it has revolved in the positive direction, it has traced out the positive angle whose measure is $+225°$.

If it has revolved in the negative direction, it has traced out the negative angle $-135°$.

Again, suppose we only know that the revolving line is in the above position. It may have made one, two, three ... complete revolutions and then have described the positive angle $+225°$. Or again, it may have made one, two, three ... complete revolutions in the negative direction and then have described the negative angle $-135°$.

In the first case, the angle it has described is either $225°$, or $360° + 225°$, or $2 \times 360° + 225°$, or $3 \times 360° + 225°$... *i.e.* $225°$ or $585°$, or $945°$, or $1305°$...

In the second case, the angle it has described is $-135°$, or $-360° - 135°$, or $-2 \times 360° - 135°$, or $-3 \times 360° - 135°$... *i.e.* $-135°$, or $-495°$, or $-855°$, or $-1215°$...

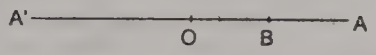
48. Positive and Negative Lines. Suppose that a man is told to start from a given milestone on a straight road and to walk 1000 yards along the road and then to stop. Unless we are told the *direction* in which he started,

we do not know his position when he stops. All we know is that he is either at a distance 1000 yards on one side of the milestone or at the same distance on the other side.

In measuring distances along a straight line it is therefore convenient to have a standard direction; this direction is called the positive direction and all distances measured along it are said to be positive. The opposite direction is called the negative direction, and all distances measured along it are said to be negative.

The standard or positive direction for lines drawn parallel to the foot of the page is towards the right.

The length OA is in the positive direction. The length OA' is in the negative direction. If the magnitude of the distance OA or OA' be a, the point A is at a distance $+a$ from O and the point A' is at a distance $-a$ from O.



All lines measured to the right have then the positive sign prefixed; all lines to the left have the negative sign prefixed.

If a point start from O and describe a positive distance OA, and then a distance AB back again toward O, equal numerically to b, the total distance it has described measured in the positive direction is $OA + AB$,

$$i.e. + a + (-b)\ i.e.\ a - b.$$

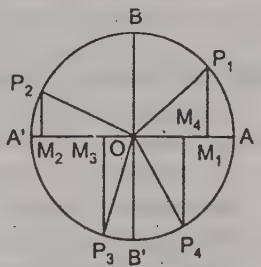
49. For lines at right angles to AA', the positive direction is from O towards the top of the page *i.e.* the direction of OB (fig., Art. 47). All lines measured from O towards the foot of the page, *i.e.* in the direction OB', are negative.

50. *Trigonometrical ratios for an angle of any magnitude.*

Let OA be the initial line (drawn in the positive direction) and let OA' be drawn in the opposite direction to OA.

Let BOB' be a line at right angles to OA, its positive direction being OB.

Let a revolving line OP start from OA and revolving in either direction, positive or negative, trace out an angle of any magnitude whatever. From a point P in the revolving line draw PM perpendicular to AOA'.



[Four positions of the revolving line are given in the figure, one in each of the four quadrants and the suffixes 1, 2, 3 and 4 are attached to P for the purpose of distinction.]

We then have the following definitions, which are the same as those given in Art. 23 for the simple case of an acute angle.

$\frac{MP}{OP}$ is called the **Sine** of the angle AOP

$\frac{OM}{OP}$ is called the **Cosine** of the angle AOP

$\frac{MP}{OM}$ is called the **Tangent** of the angle AOP

$\frac{OM}{MP}$ is called the **Cotangent** of the angle AOP

$\frac{OP}{OM}$ is called the **Secant** of the angle AOP

$\frac{OP}{MP}$ is called the **Cosecant** of the angle AOP

The quantities $1 - \cos AOP$ and $1 - \sin AOP$ are respectively called the **Versed Sine** and the **Coversed Sine** of AOP

51. In exactly the same manner as in Art. 27 it may be shown that, for all values of the angle $AOP\ (= \theta)$, we have

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta,$$

$$\sec^2\theta = 1 + \tan^2\theta$$

and

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$$

52. Signs of the trigonometrical ratios.

First quadrant. Let the revolving line be in the first quadrant, as OP_1. This revolving line is always positive. Here OM_1 and M_1P_1 are both positive, so that all the trigonometrical ratios are then positive.

Second quadrant. Let the revolving line be in the second quadrant, as OP_2. Here M_2P_2 is positive and OM_2 is negative.

The sine, being equal to the ratio of a positive quantity to a positive quantity, is therefore positive.

The cosine, being equal to the ratio of a negative quantity to a positive quantity, is therefore negative.

The tangent, being equal to the ratio of a positive quantity to a negative quantity, is therefore negative.

The cotangent is negative.

The secant is negative.

The cosecant is positive.

Third quadrant. If the revolving line be, as OP_3, in the third quadrant, we have both M_3P_3 and OM_3 negative.

The sine is therefore negative.

The cosine is negative.

The tangent is positive.

The cotangent is positive.

The secant is negative.

The cosecant is negative.

Fourth quadrant. Let the revolving line be in the fourth quadrant, as OP_4. Here M_4P_4 is negative and OM_4 is positive.

The sine is therefore negative.
The cosine is positive.
The tangent is negative.
The cotagent is negative.
The secant is positive.
The cosecant is negative.

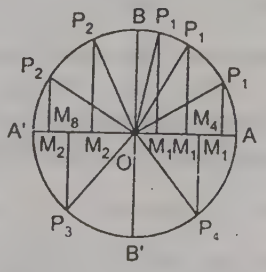
The annexed table shews the signs of the trigonometrical ratios according to the quadrant in which lies the revolving line which bounds the angle considered.

	B		
sin +			sin +
cos −			cos +
tan −			tan +
cot −			cot +
sec −			sec +
A′ cosec +			cosec + A
	O		
sin −			sin −
cos −			cos +
tan +			tan −
cot +			cot −
sec −			sec +
cosec −			cosec −
	B′		

53. *Tracing of the changes in the sign and magnitude of the trigonometrical ratios of an angle, as the angle increases from* 0° *to* 360°

Let the revolving line OP be of constant length a.

When it coincides with OA, the length OM_1 is equal to a; and, when it coincides with OB, the point M_1 coincides with O and OM_1 vanishes. Also, as the revolving line turns from OA to OB, the distance OM_1 decreases from a to zero.



Whilst the revolving line is in the second quadrant and is revolving from OB to OA', the distance OM_2 is negative and increases numerically from 0 to a [*i.e.* it *decreases algebraically* from 0 to $-a$].

In the third quadrant, the distance OM_3 *increases* algebraically from $-a$ to 0; and, in the fourth quadrant, the distance OM_4 increases from 0 to a.

In the first quadrant, the length M_1P_1 increases from 0 to a; in the second quadrant, M_2P_2 decreases from a to 0; in the third quadrant, M_3P_3 decreases algebraically from 0 to $-a$; whilst in the fourth quadrant M_4P_4 increases algebraically from $-a$ to 0.

54. Sine. In the first quadrant, as the angle increases from 0 to 90°, the sine, *i.e.* $\frac{M_1P_1}{a}$, increases from $\frac{0}{a}$ to $\frac{a}{a}$, *i.e.* from 0 to 1.

In the second quadrant, as the angle increases from 90° to 180°, the sine decreases from $\frac{a}{a}$ to $\frac{0}{a}$, *i.e.* from 1 to 0.

In the third quadrant, as the angle increases from 180° to 270°, the sine *decreases* from $\frac{0}{a}$ to $\frac{-a}{a}$, *i.e.* from 0 to -1.

In the fourth quadrant, as the angle increases from 270° to 360°, the sine *increases* from $\frac{-a}{a}$ to $\frac{0}{a}$, *i.e.* from -1 to 0.

55. Cosine. In the first quadrant the cosine, which is equal to $\frac{OM_1}{a}$, decreases from $\frac{a}{a}$ to $\frac{0}{a}$, *i.e.* from 1 to 0.

In the second quadrant, it decreases from $\frac{0}{a}$ to $\frac{-a}{a}$, *i.e.* from 0 to -1.

In the third quadrant, it increases from $\frac{-a}{a}$ to $\frac{0}{a}$, *i.e.* from -1 to 0.

In the fourth quadrant, it increases from $\frac{0}{a}$ to $\frac{a}{a}$, *i.e.* from 0 to 1.

56. Tangent. In the first quadrant, M_1P_1 increases from 0 to a and OM_1 decreases from a to 0, so that $\frac{M_1P_1}{OM_1}$, continually increases (for its numerator continually increases and its denominator continually decreases).

When OP_1 coincides with OA, the tangent is 0; when the revolving line has turned through an angle which is slightly less than a right angle, so that OP_1 nearly coincides with OB, then M_1P_1 is very nearly equal to a and OM_1 is very small. The ratio $\frac{M_1P_1}{OM_1}$ is therefore very large, and the nearer OP_1 gets to OB the larger does the ratio become, so that, by taking the revolving line near enough to OB, we can make the tangent as large as we

please. This is expressed by saying that, when the angle is equal to 90°, its tangent is infinite.

The symbol ∞ is used to denote an infinitely great quantity.

Hence in the first quadrant the tangent increases from 0 to ∞.

In the second quadrant, when the revolving line has described an angle AOP_2 slightly greater than a right angle, M_2P_2 is very nearly equal to a and OM_2 is very small and negative, so that the corresponding tangent is very large and negative.

Also, as the revolving line turns from OB to OA', M_2P_2 decreases from a to 0 and OM_2 is negative and decreases from 0 to $-a$, so that when the revolving line coincides with OA' the tangent is zero.

Hence, in the second quadrant, the tangent increases from $-\infty$ to 0.

In the third quadrant, both M_3P_3 and OM_3 are negative, and hence their ratio is positive. Also, when the revolving line coincides with OB', the tangent is infinite.

Hence, in the third quadrant, the tangent increases from 0 to ∞.

In the fourth quadrant, M_4P_4 is negative and OM_4 is positive, so that their ratio is negative. Also, as the revolving line passes through OB' the tangent changes from $+\infty$ to $-\infty$[just as in passing through OB].

Hence in the fourth quadrant, the tangent increases from $-\infty$ to 0.

57. Cotangent. When the revolving line coincides with OA, M_1P_1 is very small and OM_1 is very nearly equal to a, so that the cotangent, *i.e.* the ratio $\frac{OM_1}{M_1P_1}$ is infinite to start with. Also, as the revolving line rotates from OA to OB, the quantity M_1P_1 increases from 0 to a and OM_1 decreases from a to 0.

Hence, in the first quadrant, the contangent decreases from ∞ to 0.

In the second quadrant, M_2P_2 is positive and OM_2 negative, so that the cotangent decreases from 0 to $\frac{-a}{0}$, *i.e.* from 0 to $-\infty$.

In the third quadrant, it is positive and decreases from ∞ to 0 [for as the revolving line crosses OA' the cotangent changes from $-\infty$ to ∞].

In the fourth quadrant, it is negative and decreases from 0 to $-\infty$.

58. Secant. When the revolving line coincides with OA the value of OM_1 is a, so that the value of the secant is then unity.

As the revolving line turns from OA to OB, OM_1 decreases from a to 0, and when the revolving line coincides with OB the value of the secant is $\frac{a}{0}$, *i.e.* ∞.

Hence, in the first quadrant, the secant increases from 1 to ∞.

In the second quadrant, OM_2 is negative and decreases from 0 to $-a$. Hence, in this quadrant, the secant increases from $-\infty$ to -1 [for as

the revolving line crosses OB the quantity OM_1 changes sign and therefore the secant changes from $+\infty$ to $-\infty$].

In the third quadrant, OM_3 is always negative and increases from $-a$ to 0; therefore the secant decreases from -1 to $-\infty$.

In the fourth quadrant, OM_4 is always positive and increases from 0 to a. Hence, in this quadrant, the secant decreases from ∞ to $+1$.

59. Cosecant. The change in the cosecant may be traced in a similar manner to that in the secant.

In the first quadrant, it decreases from ∞ to $+1$.

In the second quadrant, it increases from $+1$ to $+\infty$.

In the third quadrant, it increases from $-\infty$ to -1.

In the fourth quadrant, it decreases from -1 to $-\infty$.

60. The foregoing results are collected in the annexed table.

B

In the second quadrant, the			In the first quadrant, the		
sine	decreases from	1 to 0	sine	increases from	0 to 1
cosine	decreases from	0 to -1	cosine	decreases from	1 to 0
tangent	increases from	$-\infty$ to 0	tangent	increases from	0 to ∞
cotangent	decreases from	0 to $-\infty$	contangent	decreases from	∞ to 0
secant	increases from	$-\infty$ to -1	secant	increases from	1 to ∞
cosecant	increases from	1 to ∞	cosecant	decreases from	∞ to 1

A′ O A

In the third quadrant, the			In the fourth quadrant, the		
sine	decreases from	0 to -1	sine	increases from	-1 to 0
cosine	increases from	-1 to 0	cosine	increases from	0 to 1
tangent	increases from	0 to ∞	tangent	increases from	$-\infty$ to 0
cotangent	decreases from	∞ to 0	contangent	decreases from	0 to $-\infty$
secant	decreases from	-1 to $-\infty$	secant	decreases from	∞ to 1
cosecant	increases from	$-\infty$ to -1	cosecant	decreases from	-1 to $-\infty$

B′

61. Periods of the trigonometrical functions.

As an angle increases from 0 to 2π radians, *i.e.* whilst the revolving line makes a complete revolution, its sine first increases from 0 to 1, then decreases from 1 to -1, and finally increases from -1 to 0, and thus the sine goes through all its changes, returning to its original value.

Similarly, as the angle increases from 2π radians to 4π radians, the sine goes through the same series of changes.

Also, the sines of any two angles which differ by four right angles, *i.e.* 2π radians, are the same.

This is expressed by saying that the **period of the sine is 2π**

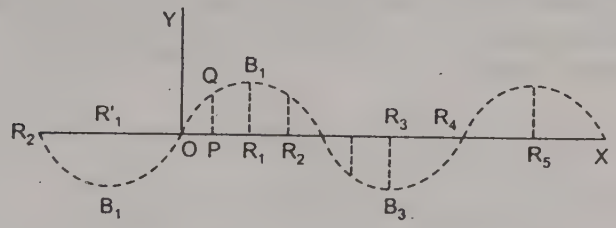
Similarly, the cosine, secant, and cosecant go through all their changes as the angle increases by 2π.

The tangent, however, goes through all its changes as the angle increases from 0 to π radians, *i.e.* whilst the revolving line turns through two right angles. Similarly for the contangent.

The period of the sine, cosine, secant and cosecant is therefore 2π radians; the period of the tangent and cotangent is π radians.

Since the values of the trigonometrical functions repeat over and over again as the angle increases, they are called **periodic functions.**

***62.** The variations in the values of the trigonometrical ratios may be graphically represented to the eye by means of curves constructed in the following manner.



Sine-Graph.

Let OX and OY be two straight lines at right angles and let the magnitudes of angles be represented by lengths measured along OX.

Let $R_1, R_2, R_3, \ldots$ be points such that the distances $OR_1, R_1R_2, R_2R_3, \ldots$ are equal. If then the distance OR_1 represent a right angle, the distances $OR_2, OR_3, OR_4, \ldots$ must represent two, three, four, ... right angles.

Also, if P by any point on the line OX, then OP represents an angle which bears the same ratio to a right angle that OP bears to OR_1.

[For example, if OP be equal to $\frac{1}{3}OR_1$, then OP would represent one-third of a right angle; if P bisected R_3R_4, then OP would represent $3\frac{1}{2}$ right angles.]

Let also OR_1 be so chosen that one unit of length represents one radian; since OR_2 represents two right angles *i.e.* π radians, the length OR_2 must be π units of length, *i.e.* about $3\frac{1}{7}$ units of length.

In a similar manner, negative angles are represented by distances OR_1', OR_2', ... measured from O in a negative direction.

At each point P erect a perpendicular PQ to represent the sine of the angle which is represented by OP; if the sine be positive, the perpendicular is to be drawn parallel to OY in the positive direction; if the sine be negative, the line is to be drawn in the negative direction.

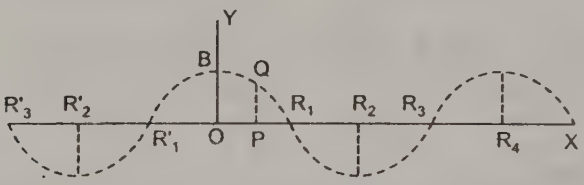
[For example, since OR_1 represents a right angle, the sine of which is 1, we erect a perpendicular R_1B_1 equal to one unit of length; since OR_2 represents an angle equal to two right angles, the sine of which is zero, we

erect a perpendicular of length zero; since OR_3 represents three right angles, the sine of which is -1, we erect a perpendicular equal to -1. *i.e.* we draw R_3B_3 downward and equal to a unit of length; if OP were equal to one-third of OR_1, it would represent $\frac{1}{3}$ of a right angle., *i.e.* 30°, the sine of which is $\frac{1}{2}$, and so we should erect a perpendicular PQ equal to one-half the unit of length.]

The ends of all these lines, thus drawn, would be found to lie on a curve similar to the one drawn above.

It would be found that the curve consisted of portions, similar to $OB_1R_2B_3R_4$, placed side by side. This corresponds to the fact that each time the angle increases by 2π, the sine repeats the same value.

***63. Cosine-Graph.**



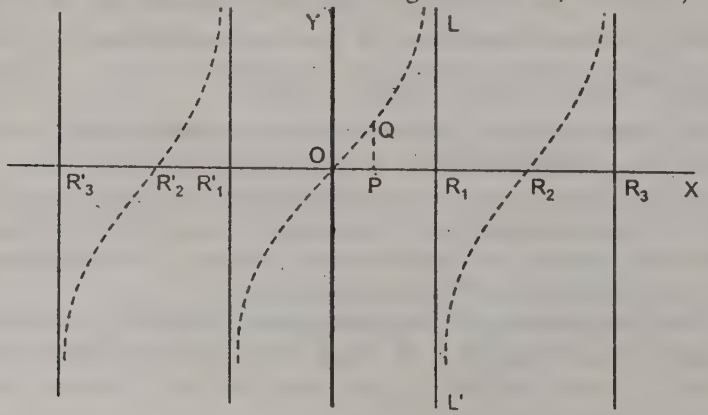
The Cosine-Graph is obtained in the same manner as the Sine-Graph, except that in this case the perpendicular PQ represents the cosine of the angle represented by OP.

The curve obtained is the same as that of Art. 62 if in that curve we move O to R_1 and let OY be drawn along R_1B_1.

***64. Tangent-Graph.**

In this case, since the tangent of a right angle is infinite and since OR_1 represents a right angle, the perpendicular drawn at R_1 must be of infinite length and the dotted curve will only meet the line R_1L at an infinite distance.

Since the tangent of an angle slightly greater than a right angle is negative and almost infinitely great, the dotted curve immediately beyond LR_1L' commences at an infinite distance on the negative side of, *i.e.* below, OX.

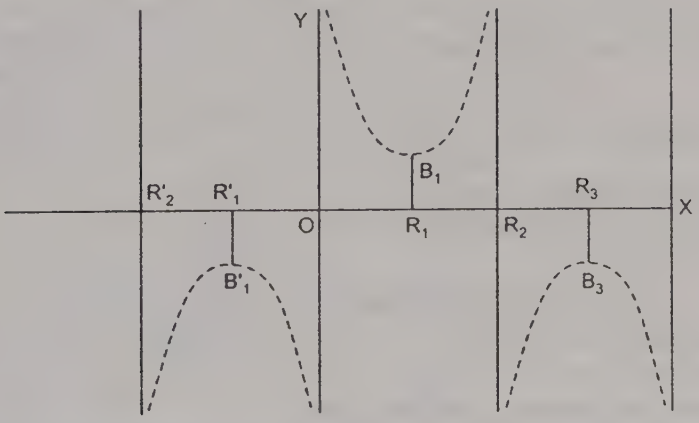


The Tangent-Graph will clearly consist of an infinite number of similar but disconnected portions, all ranged parallel to one another. Such a curve is called a Discontinuous Curve. Both the Sine-Graph and the Cosine-Graph are, on the other hand, Continuous Curves.

***65. Cotangent-Graph.** If the curve to represent the cotangent be drawn in a similar manner, it will be found to meet OY at an infinite distance above O; it will pass through the point R_1 and touch the vertical line through R_2 at an infinite distance on the negative side of OX. Just beyond R_2 it will start at an infinite distance above R_2, and proceed as before.

The curve is therefore discontinuous and will consist of an infinite number of portions all ranged side by side.

***66. Cosecant-Graph**



When the angle is zero, the sine is zero, and the cosecant is therefore infinite.

Hence the curve meets OY at infinity.

When the angle is a right angle, the cosecant is unity, and hence $R_1 B_1$ is equal to the unit of length.

When the angle is equal to two right angles its cosecant is infinity, so that the curve meets the perpendicular through R_2 at an infinite distance.

Again, as the angle increases from slightly less to slightly greater than two right angles, the cosecant changes from $+\infty$ to $-\infty$.

Hence just beyond R_2 the curve commences at an infinite distance on the negative side of, *i.e.* below, OX.

***67. Secant-Graph.** If, similarly, the Secant-Graph be traced, it will be found to be the same as the Cosecant-Graph would be if we moved OY to $R_1 B_1$.

[Some further examples of graphs will be found on pages 105, 106, 116, 207 and 208.]

MISCELLANEOUS EXAMPLES IX.

1. In a triangle one angle contains as many grades as another contains degrees, and the third contains as many centesimal seconds as there are sexagesimal seconds in the sum of the other two; find the number of radians in each angle.
2. Find the number of degrees, minutes, and seconds in the angle at the centre of a circle, whose radius is 5 m, which is subtended by an arc of length 6 m.
3. To turn radians into seconds, prove that we must multiply by 206265 nearly, and to turn seconds into radians the multiplier must be ·0000048.
4. If $\sin\theta$ equal $\dfrac{x^2-y^2}{x^2+y^2}$, find the values of $\cos\theta$ and $\cot\theta$
5. If $\sin\theta = \dfrac{m^2+2mn}{m^2+2mn+2n^2}$,

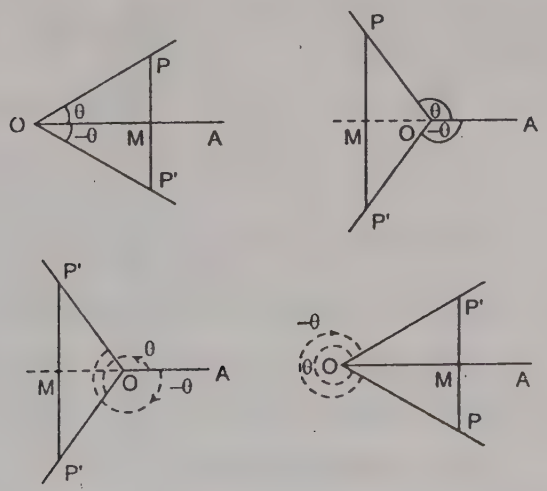
   prove that $\tan\theta = \dfrac{m^2+2mn}{2mn+2n^2}$.
6. If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$,
   prove that $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$.
7. Prove that
   $$\operatorname{cosec}^6 a - \cot^6 a = 3\operatorname{cosec}^2 a \cot^2 a + 1.$$
8. Express $2\sec^2 A - \sec^4 A - 2\operatorname{cosec}^2 A + \operatorname{cosec}^4 A$
   in terms of $\tan A$.
9. Solve the equation $3\operatorname{cosec}^2\theta = 2\sec\theta$.
10. A man on a cliff observes a boat at an angle of depression of 36°, which is making for the shore immediately beneath him. Three minutes later the angle of depression of the boat is 60°. How soon will it reach the shore ?
11. Prove that the equation $\sin\theta = x + \dfrac{1}{x}$ is impossible if x be real.
12. Show that the equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is only possible when $x = y$.

CHAPTER V

TRIGONOMETRICAL FUNCTIONS OF ANGLES OF ANY SIZE AND SIGN

[On a first reading of the subject, the student is recommended to confine his attention to the first of the four figures given in Arts. 68, 69, 70, and 72.]

68. *To find the trigonometrical ratios of an angle* $(-\theta)$ *in terms of those of* θ*, for all values of* θ*.*



Let the revolving line, starting from OA, revolve through any angle θ and stop in the position OP.

Draw PM perpendicular to OA (or OA produced) and produce it to P', so that the lengths of PM and MP' are equal.

In the geometrical triangles MOP and MOP' we have the two sides OM and MP equal to the two sides OM and MP', and the included angles OMP and OMP' are right angles.

Hence the magnitudes of the angles MOP and MOP' are the same, and OP is equal to OP'.

In each of the four figures, the magnitudes of the angle AOP (measured counter-clockwise) and of the angle AOP' (measured clockwise) are the same.

Hence the angle AOP' (measured clockwise) is denoted by $-\theta$.

Also MP and MP' are equal in magnitude but are opposite in sign. (Art. 49.) We have therefore

$$\sin(-\theta) = \frac{MP'}{OP'} = \frac{-MP}{OP} = -\sin\theta,$$

$$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta,$$

$$\tan(-\theta) = \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan\theta,$$

$$\cot(-\theta) = \frac{OM}{MP'} = \frac{OM}{-MP} = -\cot\theta$$

$$\operatorname{cosec}(-\theta) = \frac{OP'}{MP'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta$$

and $$\sec(-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\theta.$$

[In this article, and the following articles, the values of the last four trigonometrical ratios may be found, without reference to the figure, from the values of the first two ratios.

Thus $$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

$$\operatorname{cosec}(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\operatorname{cosec}\theta,$$

and $$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta.]$$

Exs. $$\sin(-30°) = -\sin 30° = -\frac{1}{2},$$

$$\tan(-60°) = -\tan 60° = -\sqrt{3},$$

and $$\cos(-45°) = \cos 45° = \frac{1}{\sqrt{2}}.$$

69. *To find the trigonometrical ratios of the angle* $(90° - \theta)$ *in terms of those of* θ, *for all values of* θ.

The relations have already been discussed in Art. 39, for values of θ less than a right angle.

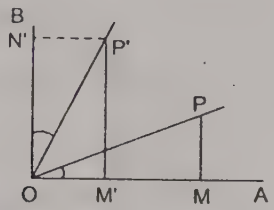
Let the revolving line, starting from OA, trace out any angle AOP denoted by θ.

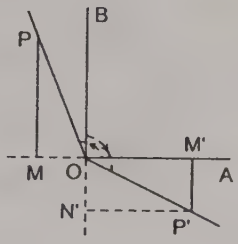
To obtain the angle $90° - \theta$, let the revolving line rotate to B and then rotate from B in the opposite direction through the angle θ, and let the position of the revolving line be then OP'.

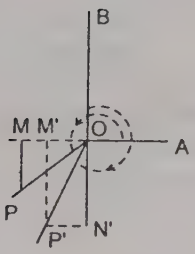
The angle AOP' is then $90° - \theta$.

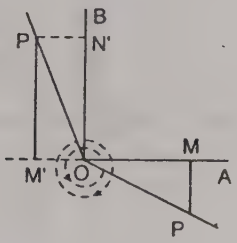
Take OP' equal to OP, and draw $P'M'$ and PM perpendicular to OA, produced if necessary. Also draw $P'N'$ perpendicular to OB, produced if necessary.

In each figure, the angles AOP and BOP' are numerically equal, by constructions.









Hence, in each figure,

$$\angle MOP = \angle N'OP' = \angle OP'M',$$

since ON' and $M'P'$ are parallel.

Hence the triangles MOP and $M'P'O$ are equal in all respects, and therefore

$OM = M'P'$ numerically, and $OM' = MP$ numerically.

Also, in each figure, OM and $M'P'$ are of the same sign, and so also are MP and OM',

i.e. $OM = +M'P'$ and $OM' = +MP$.

Hence

$$\sin(90° - \theta) = \sin AOP' = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta,$$

$$\cos(90° - \theta) = \cos AOP' = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin\theta,$$

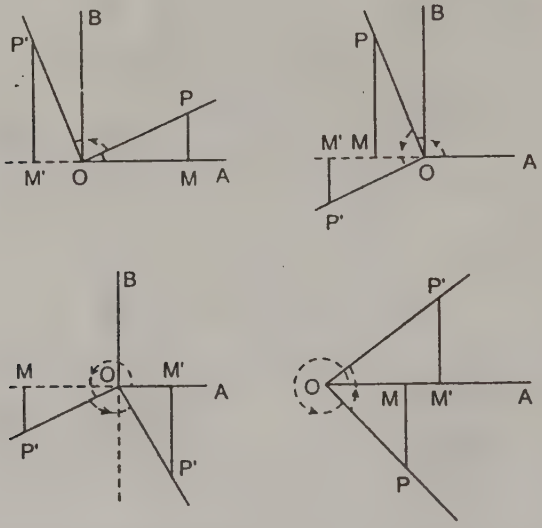
$$\tan(90° - \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot\theta,$$

$$\cot(90° - \theta) = \cot AOP' = \frac{OM'}{M'P'} = \frac{MP}{OM} = \tan\theta,$$

$$\sec(90° - \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{MP} = \operatorname{cosec}\theta,$$

and $$\operatorname{cosec}(90° - \theta) = \operatorname{cosec} AOP' = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta,$$

70. *To find the trigonometrical ratios of the angle* $(90° + \theta)$ *in terms of those of* θ, *for all values of* θ.



Let the revolving line, starting from OA, trace out any angle θ and let OP be the position of the revolving line then, so that the angle AOP is θ.

Let the revolving line turn through a right angle from OP in the positive direction to the position OP' so that the angle AOP' is $(90° + \theta)$.

Take OP' equal to OP and draw PM and $P'M'$ perpendicular to AO, produced if necessary. In eash figure, since POP' is a right angle, the sum of the angles MOP and $P'OM'$ is always a right angle.

Hence $\angle MOP = 90° - \angle P'OM' = \angle OP'M'$

The two triangles MOP and $M'P'O$ are therefore equal in all respects.

Hence OM and $M'P'$ are numerically equal, as also MP and OM' are numerically equal.

In each figure, OM and $M'P'$ have the same sign, whilst MP and OM' have the opposite sign, so that

$$M'P' = +OM, \text{ and } OM' = -MP.$$

We therefore have

$$\sin(90° + \theta) = \sin AOP' = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta,$$

$$\cos(90° + \theta) = \cos AOP' = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\sin\theta,$$

$$\tan(90° + \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\cot\theta,$$

$$\cot(90° + \theta) = \cot AOP' = \frac{OM'}{M'P'} = \frac{-MP}{OM} = -\tan\theta,$$

$$\sec(90° + \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta,$$

and $$\operatorname{cosec}(90° + \theta) = \operatorname{cosec} AOP' = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta,$$

Exs. $\sin 150° = \sin(90° + 60°) = \cos 60° = \frac{1}{2}$,

$$\cos 135° = \cos(90° + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}},$$

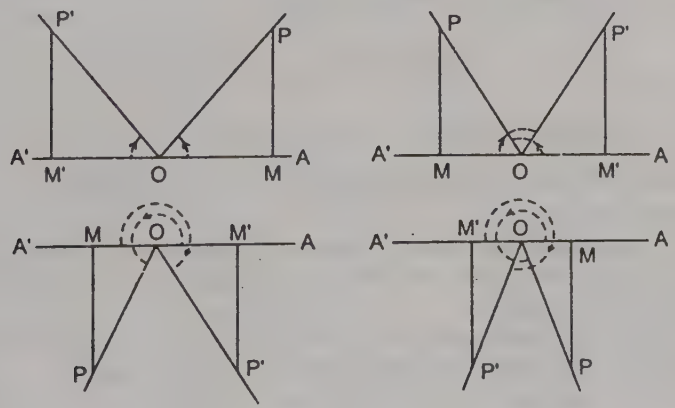
and $\tan 120° = \tan(90° + 30°) = -\cot 30° = -\sqrt{3}$

71. Supplementary Angles.

Two angles are said to be supplementary when their sum is equal to two right angles, *i.e.* the supplement of any angle θ is $180° - \theta$.

Exs. The supplement of $30° = 180° - 30° = 150°$.
The supplement of $120° = 180° - 120° = 60°$.
The supplement of $275° = 180° - 275° = -95°$.
The supplement of $-126° = 180° - (-126°) = 306°$.

72. *To find the values of the trigonometrical ratios of the angle* $(180° - \theta)$ *in terms of those of the angle* θ, *for all values of* θ.



Let the revolving line start from OA and describe any angle AOP $(= \theta)$.

To obtain the angle $180° - \theta$, let the revolving line start from OA and, after revolving through two right angles (*i.e.* into the position OA'), then revolve back through an angle θ into the position OP', so that the angle $A'OP'$ is equal in magnitude but opposite in sign to the angle AOP.

The angle AOP' is then $180° - \theta$.

Take OP' equal to OP, and draw $P'M'$ and PM perpendicular to AOA'.

The angles MOP and $M'OP'$ are equal and hence the triangles MOP and $M'OP'$ are equal in all respects.

Hence OM and OM' are equal in magnitude and so also are MP and $M'P'$.

In each figure, OM and OM' are drawn in opposite directions, whilst MP and $M'P'$ are drawn in the same direction, so that

$$OM' = -OM \text{ and } M'P' = +MP$$

Hence we have

$$\sin(180° - \theta) = \sin AOP' = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin\theta,$$

$$\cos(180° - \theta) = \cos AOP' = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\theta,$$

$$\tan(180° - \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan\theta,$$

$$\cot(180° - \theta) = \cot AOP' = \frac{OM'}{M'P'} = \frac{-OM}{MP} = -\cot\theta,$$

$$\sec(180° - \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\theta,$$

and $$\operatorname{cosec}(180° - \theta) = \operatorname{cosec} AOP' = \frac{OP'}{M'P'} = \frac{OP}{MP} = \operatorname{cosec}\theta$$

Exs. $\sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$,

$$\cos 135° = \cos(180° - 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}.$$

and $$\tan 150° = \tan(180° - 30°) = -\tan 30° = -\frac{1}{\sqrt{3}}$$

73. *To find the trigonometrical ratios of* $(180° + \theta)$ *in terms of those of* θ, *for all values of* θ.

The required relations may be obtained geometrically, as in the previous articles. The figures for this proposition are easily obtained and are left as an example for the student.

They may be deducted from the results of Art. 70, which have been proved true for all angles. For, putting $90° + \theta = B$, we have

$$\sin(180° + \theta) = \sin(90° + B) = \cos B \qquad \text{(Art. 70)}$$

and $$= \cos(90° + \theta) = -\sin\theta, \qquad \text{(Art. 70)}$$

$$\cos(180° + \theta) = \cos(90° + B) = -\sin B \qquad \text{(Art. 70)}$$

$$= -\sin(90° + \theta) = -\cos\theta, \qquad \text{(Art. 70)}$$

So $$\tan(180° + \theta) = \tan(90° + B) = -\cot B$$

$$= -\cot(90° + \theta) = \tan\theta,$$

and similarly $\cot(180° + \theta) = \cot\theta$,

$$\sec(180° + \theta) = \sec\theta,$$

and $$\operatorname{cosec}(180° + \theta) = -\operatorname{cosec}\theta.$$

74. *To find the trigonometrical ratios of an angle* $(360° + \theta)$ *in terms of those of* θ*, for all values of* θ.

In whatever position the revolving line amy be when it has described any angle θ, it will be in exactly the same position when it has made one more complete revolution in the positive direction *i.e.* when it has described an angle $360° + \theta$.

Hence the trigonometrical ratios for an angle $360° + \theta$ are the same as those for θ.

It follows that the addition or subtraction of 360°, or any multiple of 360°, to or from any angle does not alter its trigonometrical ratios.

75. From the theorems of this chapter it follows that the trigonometrical ratios of any angle whatever can be reduced to the determination of the trigonometrical ratios of an angle which lies between 0° and 45°.

For example,

$$\sin 1765° = \sin[4 \times 360° + 325°] = \sin 325° \quad \text{(Art. 74)}$$
$$= \sin(180° + 145°) = -\sin 145° \quad \text{(Art. 73)}$$
$$= -\sin(180° - 35°) = -\sin 35° \quad \text{(Art. 72);}$$
$$\tan 1190° = \tan(3 \times 360° + 110°) = \tan 110° \quad \text{(Art. 74)}$$
$$= \tan(90° + 20°) = -\cot 20° \quad \text{(Art. 70);}$$

and $$\operatorname{cosec}(-1465°) = -\operatorname{cosec} 1465° \quad \text{(Art. 68)}$$
$$= -\operatorname{cosec}(4 \times 360° + 25°) = -\operatorname{cosec} 25° \quad \text{(Art. 74)}$$

Similarly any other such large angles may be treated. First, multiples of 360° should be subtracted until the angle lies between 0° and 360°; if it be then greater than 180°, it should be reduced by 180°; if then greater than 90°, the formulae of Art. 70 should be used, and finally, if necessary, the formulae of Art. 69 applied.

76. *The table of Art. 40 may now be extended to some important angles greater than a right angle.*

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
Cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
Cosecant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
Secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

EXAMPLES. X.

Prove that :

1. $\sin 420° \cos 390° + \cos(-300°) \sin(-330°) = 1$.
2. $\cos 570° \sin 510° - \sin 330° \cos 390° = 0$
3. $\tan 225° \cot 405° + \tan 765° \cos 675° = 0$

What are the values of $\cos A - \sin A$ and $\tan A + \cot A$ when A has the values

4. $\frac{\pi}{3}$, 5. $\frac{2\pi}{3}$, 6. $\frac{5\pi}{4}$ 7. $\frac{7\pi}{4}$ and 8. $\frac{11\pi}{3}$?

What values between 0° and 360° may A have when

9. $\sin A = \frac{1}{\sqrt{2}}$, 10. $\cos A = -\frac{1}{2}$, 11. $\tan A = -1$,
12. $\cot A = -\sqrt{3}$, 13. $\sec A = -\frac{2}{\sqrt{3}}$ and 14. $\operatorname{cosec} A = -2$?

Express in terms of the ratios of a positive angle, which is less than 45°, the quantities

15. $\sin(-65°)$. 16. $\cos(-84°)$. 17. $\tan 137°$.
18. $\sin 168°$. 19. $\cos 287°$. 20. $\tan(-246°)$.
21. $\sin 843°$. 22. $\cos(-928°)$. 23. $\tan 1145°$.
24. $\cos 1410°$ 25. $\cot(-1054°)$ 26. sec 1327° and
27. $\operatorname{cosec}(-756°)$.

What sign has $\sin A + \cos A$ for the following values of A ?

28. 140° 29. 278° 30. − 356° and 31. − 1125°.

What sign has $\sin A - \cos A$ for the following values of A ?

32. 215° 33. 825° 34. − 634° and 35. − 457°
36. Find the sines and cosines of all angles in the first four quadrants whose tangents are equal to $\cos 135°$.

Prove that

37. $\sin(270° + A) = -\cos A$, and $\tan(270° + A) = -\cot A$.
38. $\cos(270° - A) = -\sin A$, and $\cot(270° - A) = \tan A$
39. $\cos A + \sin(270° + A) - \sin(270° - A) + \cos(180° + A) = 0$.
40. $\sec(270° - A) \sec(90° - A) - \tan(270° - A) \tan(90° + A) + 1 = 0$
41. $\cot A + \tan(180° + A) + \tan(90° + A) + \tan(360° - A) = 0$.

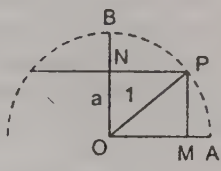
CHAPTER VI

GENERAL EXPRESSIONS FOR ALL ANGLES HAVING A GIVEN TRIGONOMETRICAL RATIO

77. *To construct the least positive angle whose sine is equal to a, where a is a proper fraction.*

Let OA be the initial line, and let OB be drawn in the positive direction perpendicular to OA.

Measure off along OB a distance ON which is equal to a units of length. [If a be negative, the point N will lie in BO produced.]



Through N draw NP parallel to OA. With centre O, and radius equal to the unit of length, describe a circle and let it meet NP in P.

Then AOP will be the required angle.

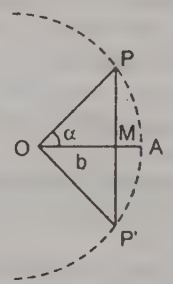
Draw PM perpendicular to OA, so that

$$\sin AOP = \frac{MP}{OP} = \frac{ON}{OP} = \frac{a}{1} = a.$$

The sine of AOP is therefore equal to the given quantity, and hence AOP is the angle required.

78. *To construct the least positive angle whose cosine is equal to b, where b is a proper fraction.*

Along the initial line measure off a distance OM equal to b and draw MP perpendicular to OA. [If b be negative, M will lie on the other side of O in the line AO produced.]



With centre O, and radius equal to unity, describe a circle and let it meet MP in P.

Then AOP is the angle required. For

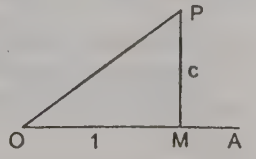
$$\cos AOP = \frac{OM}{OP} = \frac{b}{1} = b.$$

79. *To construct the least positive angle whose tangent is equal to c.*

Along the initial line measure off OM equal to unity, and erect a perpendicular MP. Measure off MP equal to c.

Then,
$$\tan AOP = \frac{MP}{OM} = c,$$

so that AOP is the required angle.



80. It is clear from the definition given in Art. 50, that, when an angle is given, so also is its sine. The converse statement is not correct; there is more than one angle having a given sine; for example, the angles 30°, 150°, 390°, $-210°$, ... all have their sine equal to $\frac{1}{2}$.

Hence, when the sine of an angle is given, we do not definitely know the angle; all we know is that the angle is one out of a large number of angles.

Similar statements are true if the cosine, tangent, or any other trigonometrical function of the angle be given.

Hence, simply to give *one* of the trigonometrical functions of an angle does not determine it without ambiguity.

81. Suppose we know that the revolving line OP coincides with the initial line OA. All we know is that the revolving line has made 0, or 1, or 2, or 3, ... complete revolutions, either positive or negative.

But when the revolving line has made one complete revolution, the angle it has described is (Art. 17) equal to 2π radians.

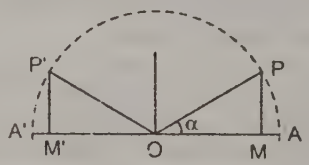
Hence, when the revolving line OP coincides with the initial line OA, the angle that it has described is 0, or 1, or 2, or 3, ... times 2π radians, in either the positive or negative directions, *i.e.* either 0, or $\pm 2\pi$, or $\pm 4\pi$, or $\pm 6\pi$, ... radians.

This is expressed by saying that when the revolving line coincides with the initial line the angle it has described is $2n\pi$, where n is some positive or negative integer.

82. Theorem. *To find a general expression to include all angles which have the same sine.* [See also Art. 102.]

Let AOP be any angle having the given sine, and let it be denoted by a.

Draw PM perpendicular to OA and produce MO to M', making OM' equal to MO, and draw $M'P'$ parallel and equal to MP.



As in Art. 72, the angle AOP' is equal to $\pi - a$.

When the revolving line is in either of the positions OP or OP', and in no other position, the sine of the angle traced out is equal to the given sine.

When the revolving line is in the position OP, it has made a whole number of complete revolutions and then described an angle α, *i.e.*, by the previous article, it has described an angle equal to

$$2r\pi + \alpha \qquad \ldots(1)$$

where r is zero or some positive or negative integer.

When the revolving line is in the position OP', it has, similarly, described an angle $2r\pi + AOP'$, *i.e.* an angle $2r\pi + \pi - \alpha$,

i.e. $$(2r + 1)\pi - \alpha \qquad \ldots(2)$$

where r is zero or some positive or negative integer.

All these angles will be found to be included in the expression.

$$n\pi + (-1)^n \alpha \qquad \ldots(3)$$

where n is zero or a positive or negative integer.

For, when $n = 2r$, since $(-1)^{2r} = +1$, the expression (3) gives $2r\pi + \alpha$, which is the same as the expression (1).

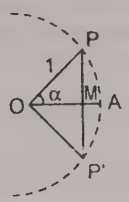
Also, when $n = 2r + 1$, since $(-1)^{2r+1} = -1$, the expression (3) gives $(2r + 1)\pi - \alpha$, which is the same as the expression (2).

Cor. Since all angles which have the same sine have also the same cosecant, the expression (3) includes all angles which have the same cosecant as α.

83. Theorem. *To find a general expression to include all angles which have the same cosine.* [See also Art. 103]

Let AOP be any angle having the given cosine, and let it be denoted by α.

Draw PM perpendicular to OA and produce it to P', making MP' equal to PM.



When the revolving line is in the position OP or OP', and in no other position, then, as in Art. 78, the cosine of the angle traced out is equal to the given cosine.

When the revolving line is in the position OP, it has made a whole number of complete revolutions and then described an angle α, *i.e.* it has described an angle $2n\pi + \alpha$, where n is zero or some positive or negative integer.

When the revolving line is in the position OP', it has made a whole number of complete revolutions and then described an angle $-\alpha$, *i.e.* it has described an angle $2n\pi - \alpha$.

All these angles are included in the expression.

$$2n\pi \pm \alpha \qquad \ldots(1)$$

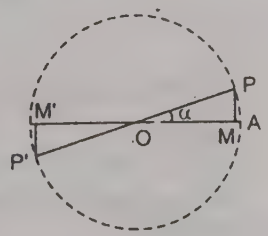
where n is zero or some positive or negative integer.

Cor. The expression (1) includes all angles having the same secant as α.

84. Theorem. *To find a general expression for all angles which have the same tangent.* [See also Art. 104]

Let AOP be any angle having the given tangent, and let it be denoted by α.

Produce PO to P', making OP' equal to OP, and draw $P'M'$ perpendicular to OM'.



As in Art. 73, the angles AOP and AOP' have the same tangent; also the angle $AOP' = \pi + \alpha$.

When the revolving line is in the position OP, it has described a whole number of complete revolutions and then turned through an angle α, *i.e.* it has described an angle

$$2r\pi + \alpha \qquad \ldots(1)$$

where r is zero or some positive or negative integer.

When the revolving line is in the position OP', it has similarly described an angle $2r\pi + (\pi + \alpha)$,

i.e. $$(2r + 1)\pi + \alpha \qquad \ldots(2)$$

All these angles are included in the expression:

$$n\pi + \alpha \qquad \ldots(3)$$

where n is zero or some positive or negative integer.

For, when n is even, $(= 2r$ say), the expression (3) gives the same angles as the expression (1).

Also, when n is odd, $(= 2r + 1$ say), it gives the same angles as the expression (2).

Cor. The expression (3) includes all angles which have the same cotangent as α.

85. In Arts. 82, 83, and 84 the angle α is any angle satisfying the given condition. In practical examples it is, in general, desirable to take α as the smallest positive angle wich is suitable.

Ex. 1. *Write down the general expression for all angles,*

(1) *whose sine is equal to* $\frac{\sqrt{3}}{2}$,

(2) *whose cosine is equal to* $-\frac{1}{2}$,

and (3) *whose tangent is equal to* $\frac{1}{\sqrt{3}}$.

(1) The smallest angle, whose sine is $\frac{\sqrt{3}}{2}$, is 60°, *i.e.* $\frac{\pi}{3}$.

Hence, by art. 82, the general expression for all the angles which have this sine is $$n\pi + (-1)^n \frac{\pi}{3}.$$

(2) The smallest positive angle, whose cosine is $-\frac{1}{2}$,

$$\text{is } 120°, \textit{i.e.} \frac{2\pi}{3}.$$

Hence by Art. 83, the general expression for all the angles which have this cosine is

$$2n\pi \pm \frac{2\pi}{3}.$$

(3) The smallest positive angle, whose tangent is $\frac{1}{\sqrt{3}}$,

$$\text{is } 30°, \textit{i.e.} \frac{\pi}{6}.$$

Hence, by Art. 84, the general expression for all the angles which have this tangent is

$$n\pi + \frac{\pi}{6}.$$

Ex. 2. *What is the most general value of θ satisfying the equation* $\sin^2\theta = \frac{1}{4}$?

Here we have $\sin\theta = \pm\frac{1}{2}$.

Taking the upper sign,

$$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$$

$$\therefore \theta = n\pi + (-1)^n\frac{\pi}{6}.$$

Taking the lower sign,

$$\sin\theta = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right).$$

$$\therefore\ \theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right).$$

Putting both solutions together, we have

$$\theta = n\pi \pm (-1)^n\frac{\pi}{6},$$

or, what is the same expression,

$$\theta = n\pi \pm \frac{\pi}{6}.$$

Ex. 3. *What is the most general value of θ which satisfies both of the equations* $\sin\theta = -\frac{1}{2}$ *and* $\tan\theta = \frac{1}{\sqrt{3}}$ *?*

Considering only angles between 0° and 360°, the only values of θ, when $\sin\theta = -\frac{1}{2}$, are 210° and 330°. Similarly, the only values of θ, when $\tan\theta = \frac{1}{\sqrt{3}}$, are 30° and 210°.

The only value of θ, between 0° and 360°, satisfying both conditions is therefore 210°, *i.e.*, $\frac{7\pi}{6}$.

The most general value is hence obtained by adding any multiple of four right angles to this angle, and hence is $2n\pi + \frac{7\pi}{6}$, where n is any positive or negative integer.

EXAMPLES. XI.

What are the most general values of θ which satisfy the equations.

1. $\sin\theta = \frac{1}{2}$. **2.** $\sin\theta = -\frac{\sqrt{3}}{2}$. **3.** $\sin\theta = \frac{1}{\sqrt{2}}$

4. $\cos\theta = -\frac{1}{2}$. **5.** $\cos\theta = \frac{\sqrt{3}}{2}$. **6.** $\cos\theta = -\frac{1}{\sqrt{2}}$

7. $\tan\theta = \sqrt{3}$. 8. $\tan\theta = -1$ 9. $\cot\theta = 1$

10. $\sec\theta = 2$. 11. $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$. 12. $\sin^2\theta = 1$

13. $\cos^2\theta = \frac{1}{4}$ 14. $\tan^2\theta = \frac{1}{3}$. 15. $4\sin^2\theta = 3$.

16. $2\cot^2\theta = \operatorname{cosec}^2\theta$. 17. $\sec^2\theta = \frac{4}{3}$?

18. What is the most general value of θ that satisfies both of the equations
$$\cos\theta = -\frac{1}{\sqrt{2}} \text{ and } \tan\theta = 1?$$

19. What is the most general value of θ that satisfies both of the equations
$$\cot\theta = -\sqrt{3} \text{ and } \operatorname{cosec}\theta = -2?$$

20. If $\cos(A - B) = \frac{1}{2}$, and $\sin(A + B) = \frac{1}{2}$, find the smallest positive values of A and B and also their most general values.

21. If $\tan(A - B) = 1$, and $\sec(A + B) = \frac{2}{\sqrt{3}}$, find the smallest positive values of A and B and also their most general values.

22. Find the angles between 0° and 360° which have respectively (1) their sines equal to $\frac{\sqrt{3}}{2}$. (2) their cosines equal to $-\frac{1}{2}$, and (3) their tangents equal to $\frac{1}{\sqrt{3}}$.

23. Taking into consideration only angles between 0° and 180°, how many values of x are there if (1) $\sin x = \frac{5}{7}$, (2) $\cos x = \frac{1}{5}$, (3) $\cos x = -\frac{4}{5}$, (4) $\tan x = \frac{2}{3}$, and (5) $\cot x = -7$?

24. Given the angle x, construct the angle y if (1) $\sin y = 2\sin x$, (2) $\tan y = 3\tan x$ (3) $\cos y = \frac{1}{2}\cos x$, and (4) $\sec y = \operatorname{cosec} x$.

25. Show that the same angles are indicated by the two following formulae : (1) $(2n - 1)\frac{\pi}{2} + (-1)^n\frac{\pi}{3}$, and (2) $2n\pi \pm \frac{\pi}{6}$, n being any integer.

26. Prove that the two formulae
$$(1)\ \left(2n + \frac{1}{2}\right)\pi \pm \alpha \text{ and } (2)\ n\pi + (-1)^n\left(\frac{\pi}{2} - \alpha\right)$$
denote the same angles, n being any integer.
Illustrate by a figure.

27. If $\theta - \alpha = n\pi + (-1)^n\beta$, prove that $\theta = 2n\pi + \alpha + \beta$ or else that $\theta = (2n + 1)\pi + \alpha - \beta$, where m and n are any integers.

28. If $\cos p\theta + \cos q\theta = 0$, prove that the different values of θ from two arithmetical progressions in which the common differences are $\frac{2\pi}{p+q}$ and $\frac{2\pi}{p-q}$ respectively.

29. Construct the angle whose sine is $\frac{3}{2+\sqrt{5}}$.

86. An equation involving the trigonometrical ratios of an unknown angle is called a trigonometrical equation.

The equation is not completely solved unless we obtain an expression for all the angles which satisfy it.

Some elementary types of equations are solved in the following article.

87. Ex.1. *Solve the equation* $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$

The equation may be written

$$2 - 2\cos^2 x + \sqrt{3}\cos x + 1 = 0,$$

i.e. $$2\cos^2 x - \sqrt{3}\cos x - 3 = 0,$$

i.e. $$(\cos x - \sqrt{3})(2\cos x + \sqrt{3}) = 0.$$

The equation is therefore satisfied by $\cos x = \sqrt{3}$, or $\cos x = -\frac{\sqrt{3}}{2}$.

Since the cosine of an angle cannot be numerically greater than unity, the first factor gives no solution.

The smallest positive angle, whose cosine is $-\frac{\sqrt{3}}{2}$, is 150°, *i.e.* $\frac{5\pi}{6}$.

Hence the most general value of the angle, whose cosine is $-\frac{\sqrt{3}}{2}$, is

$$2n\pi \pm \frac{5\pi}{6}. \qquad \text{(Art. 83)}$$

This is the general solution of the given equation.

Ex. 2. *Solve the equation* $\tan 5\theta = \cot 2\theta$.

The equation may be written

$$\tan 5\theta = \tan\left(\frac{\pi}{2} - 2\theta\right).$$

Now the most general value of the angle, that has the same tangent as $\frac{\pi}{2} - 2\theta$, is, by Art. 84, $n\pi + \frac{\pi}{2} - 2\theta$,

where n is any positive or negative integer.

The most general solution of the equation is therefore

$$5\theta = n\pi + \frac{\pi}{2} - 2\theta$$

$$\therefore \quad \theta = \frac{1}{7}\left(n\pi + \frac{\pi}{2}\right),$$

where n is any integer.

EXAMPLES. XII

Solve the equations :

1. $\cos^2\theta - \sin\theta - \frac{1}{4} = 0.$
2. $2\sin^2\theta + 3\cos\theta = 0$
3. $2\sqrt{3}\cos^2\theta = \sin\theta$
4. $\cos\theta + \cos^2\theta = 1.$
5. $4\cos\theta - 3\sec\theta = 2\tan\theta.$
6. $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$
7. $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$
8. $\cot^2\theta + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\cot\theta + 1 = 0.$
9. $\cot\theta - ab\tan\theta = a - b$
10. $\tan^2\theta + \cot^2\theta = 2$
11. $\sec\theta - 1 = (\sqrt{2} - 1)\tan\theta$
12. $3(\sec^2\theta + \tan^2\theta) = 5$
13. $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$
14. $4\cos^2\theta + \sqrt{3} = 2(\sqrt{3} + 1)\cos\theta$
15. $3\sin^2\theta - 2\sin\theta = 1.$
16. $\sin 5\theta = \frac{1}{\sqrt{2}}.$
17. $\sin 9\theta = \sin\theta.$
18. $\sin 3\theta = \sin 2\theta$
19. $\cos m\theta = \cos n\theta$
20. $\sin 2\theta = \cos 3\theta$
21. $\cos 5\theta = \cos 4\theta.$
22. $\cos m\theta = \sin n\theta.$
23. $\cot\theta = \tan 8\theta.$
24. $\cot\theta = \tan n\theta.$
25. $\tan 2\theta = \tan\frac{2}{\theta}.$
26. $\tan 2\theta \tan\theta = 1.$
27. $\tan^2 3\theta = \cot^2\alpha$
28. $\tan 3\theta = \cot\theta.$
29. $\tan^2 3\theta = \tan^2\alpha.$
30. $3\tan^2\theta = 1.$
31. $\tan mx + \cot nx = 0.$
32. $\tan(\pi\cot\theta) = \cot(\pi\tan\theta)$
33. $\sin(\theta - \phi) = \frac{1}{2}$, and $\cos(\theta + \phi) = \frac{1}{2}$.
34. $\cos(2x + 3y) = \frac{1}{2}$, and $\cos(3x + 2y) = \frac{\sqrt{3}}{2}$.
35. Find all the angles between $0°$ and $90°$ which satisfy the equation
$$\sec^2\theta\operatorname{cosec}^2\theta + 2\operatorname{cosec}^2\theta = 8$$
36. If $\tan^2\theta = \frac{5}{4}$, find versin θ and explain the double result.
37. If the coversin of an angle be $\frac{1}{3}$, find its cosine and cotangent.

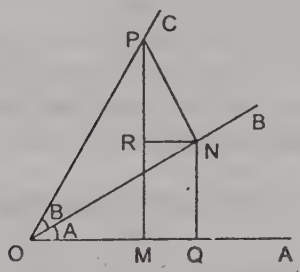
CHAPTER VII

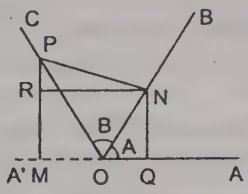
TRIGONOMETRICAL RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES

88. Theorem. *To prove that*

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

and $$\cos(A + B) = \cos A \cos B - \sin A \sin B$$





Let the revolving line start from OA and trace out the angle AOB $(= A)$, and then trace out the further angle BOC $(= B)$.

In the final position of the revolving line take any point P, and draw PM and PN perpendicular to OA and OB respectively; through N draw NR parallel to AO to meet MP in R, and draw NQ perpendicular to OA.

The angle

$$RPN = 90° - \angle PNR = \angle RNO = \angle NOQ = A$$

Hence $$\sin(A + B) = \sin AOP = \frac{MP}{OP} = \frac{MR + RP}{OP}$$

$$= \frac{QN}{OP} + \frac{RP}{OP} = \frac{QN}{ON}\frac{ON}{OP} + \frac{RP}{NP}\frac{NP}{OP}$$

$$= \sin A \cos B + \cos RPN \sin B.$$

$$\therefore \quad \sin(A + B) = \sin A \cos B + \cos A \sin B.$$

Again $$\cos(A + B) = \cos AOP = \frac{OM}{OP} = \frac{OQ - MQ}{OP}$$

$$= \frac{OQ}{OP} - \frac{RN}{OP} = \frac{OQ}{ON}\frac{ON}{OP} - \frac{RN}{NP}\frac{NP}{OP}$$

$$= \cos A \cos B - \sin RPN \sin B.$$

$$\therefore \quad \cos(A + B) = \cos A \cos B - \sin A \sin B$$

89. The figures in the previous article have been drawn only for the case in which A and B are acute angles.

The same proof will be found to apply to angles of any size, due attention being paid to the signs of the quantities involved.

The results may however be shewn to be true of all angles, without drawing any more figures, as follows.

Let A and B be acute angles, so that, by Art. 88, we know that the theorem is true for A and B.

Let $A_1 = 90° + A$, so that, by Art. 70, we have

$$\sin A_1 = \cos A, \text{ and } \cos A_1 = -\sin A.$$

Then $\sin(A_1 + B) = \sin[90° + (A + B)] = \cos(A + B)$, by art. 70,

$$= \cos A \cos B - \sin A \sin B = \sin A_1 \cos B + \cos A_1 \sin B$$

Also $\cos(A_1 + B) = \cos[90° + (A + B)] = -\sin(A + B)$

$$= -\sin A \cos B - \cos A \sin B = \cos A_1 \cos B - \sin A_1 \sin B$$

Similarly we may proceed if B be increased by 90°.

Hence the formulae of Art. 88 are true if either A or B be increased by 90°, *i.e.* they are true if the component angles lie between 0° and 180°.

Similarly, by putting $A_2 = 90° + A_1$, we can prove the truth of the theorems when either or both of the component angles have values between 0° and 270°.

By proceeding in this way, we see that the theorems are true universally.

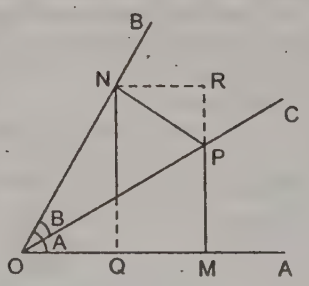
90. Theorem. *To prove that*

$$\sin(A - B) = \sin A \cos B - \cos A \sin B,$$

and

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Let the revolving line starting from the initial line OA trace out the angle AOB $(= A)$, and then, revolving in the opposite direction, trace out the angle BOC, whose magnitude is B. The angle AOC is therefore $A - B$.



Take a point P in the final position of the revolving line, and draw PM and PN perpendicular to OA and OB respectively; from N draw NQ and NR perpendicular to OA and MP respectively.

The angle

$$RPN = 90° - \angle PNR = \angle RNB = \angle QON = A.$$

Hence

$$\sin(A - B) = \sin AOC = \frac{MP}{OP} = \frac{MR - PR}{OP} = \frac{QN}{OP} - \frac{PR}{OP}$$

$$= \frac{QN}{ON}\frac{ON}{OP} - \frac{PR}{PN}\frac{PN}{OP}$$

$$= \sin A \cos B - \cos RPN \sin B,$$

so that $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

Also $\cos(A-B) = \dfrac{OM}{OP} = \dfrac{OQ+QM}{OP} = \dfrac{OQ}{OP} + \dfrac{NR}{OP}$

$$= \frac{OQ}{ON}\frac{ON}{OP} + \frac{NR}{NP}\frac{NP}{OP}$$

$$= \cos A \cos B + \sin NPR \sin B,$$

so that $\cos(A-B) = \cos A \cos B + \sin A \sin B$

91. The proofs of the previous article will be found to apply to angles of any size, provided that due attention be paid to the signs of the quantities involved.

Assuming the truth of the formulae for acute angles, we can shew them to be true universally without drawing any more figures.

For, putting $A_1 = 90° + A$, we have,

(since $\sin A_1 = \cos A$, and $\cos A_1 = -\sin A$),

$$\sin(A_1 - B) = \sin[90° + (A-B) = \cos(A-B)] \quad \text{(Art. 70)}$$

$$= \cos A \cos B + \sin A \sin B$$

$$= \sin A_1 \cos B - \cos A_1 \sin B.$$

Also $\cos(A_1 - B) = \cos[90° + (A-B)] = -\sin(A-B)$ (Art. 70)

$$= -\sin A \cos B + \cos A \sin B$$

$$= \cos A_1 \cos B + \sin A_1 \sin B.$$

Similarly we may proceed if B be increased by 90°.

Hence the theorem is true for all angles which are not greater than two right angles.

So, by putting $A_2 = 90° + A_1$, we may shew the theorems to be true for all angles less than three right angles, and so on.

Hence, by proceeding in this manner, we may shew that the theorems are true for all angles whatever.

92. The theorems of Arts. 88 and 90, which give respectively the trigonometrical functions of the sum and differences of two angles in terms of the functions of the angles themselves, are often called the **Addition and Subtraction Theorems.**

93. Ex. 1. *Find the values of* $\sin 75°$ *and* $\cos 75°$.

$$\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

and $\cos 75° = \cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Ex. 2 *Prove that* $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$,

and $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.

By Arts. 88 and 90, we have

$$\sin(A+B)\sin(A-B) = (\sin A \cos B + \cos A \sin B)$$

$$(\sin A \cos B - \cos A \sin B)$$

$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$$

$$= \sin^2 A - \sin^2 B.$$

Again, by the same articles, we have

$$\cos (A + B) \cos (A - B) = (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$$

$$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B$$

$$= \cos^2 A - \sin^2 B$$

Ex. 3. *Assuming the formulae for* $\sin (x + y)$ *and* $\cos (x + y)$, *deduce the formulae for* $\sin (x - y)$ *and* $\cos (x - y)$.

We have

$$\sin x = \sin \{(x - y) + y\} = \sin (x - y) \cos y + \cos (x - y) \sin y \quad ...(1)$$

and $$\cos x = \cos \{(x - y) + y\} = \cos (x - y) \cos y - \sin (x - y) \sin y \quad ...(2)$$

Multiplying (1) by $\cos y$ and (2) by $\sin y$ and subtracting, we have

$$\sin x \cos y - \cos x \sin y = \sin (x - y) \{\cos^2 y + \sin^2 y\} = \sin (x - y).$$

Multiplying (1) by $\sin y$ and (2) by $\cos y$ and adding, we have

$$\sin x \sin y + \cos x \cos y = \cos (x - y) \{\cos^2 y + \sin^2 y\} = \cos (x - y)$$

Hence the two formulae required are proved.

These two formulae are true for all values of the angles, since the formulae from which they are derived are true for all values of the angles.

EXAMPLES. XIII.

1. If $\sin \alpha = \frac{3}{5}$ and $\cos \beta = \frac{9}{41}$, find the value of $\sin (\alpha - \beta)$ and $\cos (\alpha + \beta)$. Verify by a graph and accurate measurement.
2. If $\sin \alpha = \frac{45}{53}$ and $\sin \beta = \frac{33}{65}$, find the value of $\sin (\alpha - \beta)$ and $\sin (\alpha + \beta)$.
3. If $\sin \alpha = \frac{15}{17}$ and $\cos \beta = \frac{12}{13}$, find the values of $\sin (\alpha + \beta)$, $\cos (\alpha - \beta)$, and $\tan (\alpha + \beta)$. Verify by a graph and accurate measurement.
4. Prove that

$$\cos (45° - A) \cos (45° - B) - \sin (45° - A) \sin (45° - B) = \sin (A + B).$$

Prove that

5. $\sin (45° + A) \cos (45° - B) + \cos (45° + A) \sin (45° - B) = \cos (A - B)$.
6. $\dfrac{\sin (A - B)}{\cos A \cos B} + \dfrac{\sin (B - C)}{\cos B \cos C} + \dfrac{\sin (C - A)}{\cos C \cos A} = 0.$
7. $\sin 105° + \cos 105° = \cos 45°$.

8. $\sin 75° - \sin 15° = \cos 105° + \cos 15°$

9. $\cos\alpha \cos(\gamma - \alpha) - \sin\alpha \sin(\gamma - \alpha) = \cos\gamma$.

10. $\cos(\alpha + \beta)\cos\gamma - \cos(\beta + \gamma)\cos\alpha = \sin\beta \sin(\gamma - \alpha)$

11. $\sin(n+1)A \sin(n-1)A + \cos(n+1)A \cos(n-1)A = \cos 2A$

12. $\sin(n+1)A \sin(n+2)A + \cos(n+1)A \cos(n+2)A = \cos A$.

94. From Arts. 88 and 90, we have, for all values of A and B,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

and $$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Hence, by addition and subtraction, we have

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B \quad ...(1)$$

and $$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B \quad ...(2)$$

From the same articles we have, for all values of A and B,

$$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$

and $$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Hence, by addition and subtraction, we have

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B \quad ...(3)$$

and $$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B \quad ...(4)$$

Put $A + B = C$, and $A - B = D$, so that

$$A = \frac{C + D}{2}, \text{ and } B = \frac{C - D}{2}.$$

On making these substitutions, the relations (1) to (4) become, for all values of C and D,

$$\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2} \quad ...(I),$$

$$\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2} \quad ...(II),$$

$$\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2} \quad ...(III),$$

and $$\cos C - \cos D = 2 \sin \frac{C + D}{2} \sin \frac{D - C}{2} \quad ...(IV).$$

[The student should carefully notice that the second factor of the right-hand member of IV is $\sin \frac{D - C}{2}$, and not $\sin \frac{C - D}{2}$.]

95. These relations I to IV are extremely important and should be very carefully committed to memory.

On account of their great importance we give a geometrical proof for the case when C and D are acute angles.

Let AOC be the angle C and AOD the angle D. Bisect the angle COD by the straight line OE. On OE take a point P and draw QPR perpendicular to OP to meet OC and OD in Q and R respectively.

Draw PL, QM, and RN perpendicular to OA and through R draw RST perpendicular to PL to QM to meet them in S and T respectively.

Since the angle DOC is $C - D$, each of the angles DOE and EOC is $\frac{C-D}{2}$, and also

$$\angle AOE = \angle AOD + \angle DOE = D + \frac{C-D}{2} = \frac{C+D}{2}.$$

Since the two triangles POR and POQ are equal in all respects, we have $OQ = OR$, and $PR = PQ$, so that

$$RQ = 2RP$$

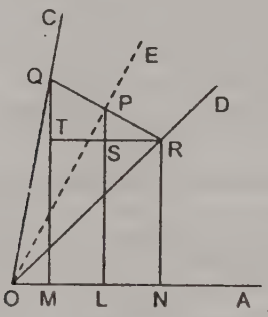
Hence $QT = 2PS$, and $RT = 2RS$, *i.e.* $MN = 2ML$.

Therefore

$$MQ + NR = TQ + 2LS = 2SP + 2LS = 2LP.$$

Also

$$OM + ON = 2OM + MN = 2OM + 2ML = 2OL.$$



Hence $$\sin C + \sin D = \frac{MQ}{OQ} + \frac{NR}{OR} = \frac{MQ + NR}{OR}$$

$$= \frac{2LP}{OR} = 2\frac{LP}{OP}\cdot\frac{OP}{OR} = 2\sin LOP \cos POR$$

$$= 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}.$$

Again, $$\sin C - \sin D = \frac{MQ}{OQ} - \frac{NR}{OR} = \frac{MQ - NR}{OR} = \frac{TQ}{OR}$$

$$= 2\frac{SP}{OR} = 2\frac{SP}{RP}\cdot\frac{RP}{OR} = 2\cos SPR \sin ROP$$

$$= 2\cos\frac{C+D}{2}\sin\frac{C-D}{2},$$

$$\left[\text{for } \angle SPR = 90° - \angle SPO = \angle LOP = \frac{C+D}{2}\right].$$

Also, $$\cos C + \cos D = \frac{OM}{OQ} + \frac{ON}{OR} = \frac{OM + ON}{OR}.$$

$$= 2\frac{OL}{OR} = 2\frac{OL}{OP}\cdot\frac{OP}{OR}$$

$$= 2\cos LOP \cos POR$$
$$= 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}.$$

Finally, $\cos D - \cos C = \frac{ON}{OR} - \frac{OM}{OQ} = \frac{ON - OM}{OR}$
$$= \frac{MN}{OR} = 2\frac{SR}{OR} = \frac{2SR}{PR}\cdot\frac{PR}{OR}$$
$$= 2\sin SPR \sin POR$$
$$= 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}.$$

96. The student is strongly urged to make himself perfectly familiar with the formulae of the previous article and to carefully practise himself in their application; perfect familiarity with these formulae will considerably facilitate his further progress.

The formulae are very useful, because they change sums and differences of certain quantities into products of certain other quantities, and products of quantities are, as the student probably knows from Algebra, easily dealt with by the help of logarithms.

We subjoin a few examples of their use.

Ex. 1. $\sin 6\theta + \sin 4\theta = 2\sin\frac{6\theta + 4\theta}{2}\cos\frac{6\theta - 4\theta}{2} = 2\sin 5\theta\cos\theta.$

Ex. 2. $\cos 3\theta - \cos 7\theta = 2\sin\frac{3\theta + 7\theta}{2}\sin\frac{7\theta - 3\theta}{2} = 2\sin 5\theta\sin 2\theta.$

Ex. 3. $$\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{2\cos\frac{75^\circ + 15^\circ}{2}\sin\frac{75^\circ - 15^\circ}{2}}{2\cos\frac{75^\circ + 15^\circ}{2}\cos\frac{75^\circ - 15^\circ}{2}}$$
$$= \frac{2\cos 45^\circ \sin 30^\circ}{2\cos 45^\circ \cos 30^\circ} = \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \cdot 57735 \ldots .$$

[This is an example of the simplification given by these formulae; it would be a very long and tiresome process to look out from the tables the values of sin 75°, sin 15°, cos 75° and cos 15°, and then to perform the division of one long decimal fraction by another.]

Ex. 4. *Simplify the expression.*

$$\frac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)}.$$

On applying the formulae of Art. 94, this expression

$$= \frac{2\sin\frac{\theta + 3\theta}{2}\sin\frac{3\theta - \theta}{2} \times 2\sin\frac{8\theta + 2\theta}{2}\cos\frac{8\theta - 2\theta}{2}}{2\cos\frac{5\theta + \theta}{2}\sin\frac{5\theta - \theta}{2} \times 2\sin\frac{4\theta + 6\theta}{2}\sin\frac{6\theta - 4\theta}{2}}$$
$$= \frac{4\,.\,\sin 2\theta \sin\theta\,.\,\sin 5\theta \cos 3\theta}{4\,.\,\cos 3\theta \sin 2\theta\,.\,\sin 5\theta \sin\theta} = 1.$$

EXAMPLES. XIV.

Prove that :

1. $\dfrac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta.$
2. $\dfrac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta} = -\tan\theta$
3. $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$
4. $\dfrac{\sin 7A - \sin A}{\sin 8A - \sin 2A} = \cos 4A \sec 5A.$
5. $\dfrac{\cos 2B + \cos 2A}{\cos 2B - \cos 2A} = \cot(A + B)\cot(A - B).$
6. $\dfrac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \dfrac{\tan(A + B)}{\tan(A - B)}.$
7. $\dfrac{\sin A + \sin 2A}{\cos A - \cos 2A} = \cot\dfrac{A}{2}.$
8. $\dfrac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A.$
9. $\dfrac{\cos 2B - \cos 2A}{\sin 2B + \sin 2A} = \tan(A - B).$
10. $\cos(A + B) + \sin(A - B) = 2\sin(45° + A)\cos(45° + B).$
11. $\dfrac{\cos 3A - \cos A}{\sin 3A - \sin A} + \dfrac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \dfrac{\sin A}{\cos 2A \cos 3A}.$
12. $\dfrac{\sin(4A - 2B) + \sin(4B - 2A)}{\cos(4A - 2B) + \cos(4B - 2A)} = \tan(A + B).$
13. $\dfrac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta \cos 4\theta.$
14. $\dfrac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \cos 2\theta - \sin 2\theta \tan 3\theta.$
15. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A.$
16. $\dfrac{\sin(\theta + \phi) - 2\sin\theta + \sin(\theta - \phi)}{\cos(\theta + \phi) - 2\cos\theta + \cos(\theta - \phi)} = \tan\theta.$
17. $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}.$
18. $\dfrac{\sin(A - C) + 2\sin A + \sin(A + C)}{\sin(B - C) + 2\sin B + \sin(B + C)} = \dfrac{\sin A}{\sin B}.$
19. $\dfrac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A.$
20. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan\dfrac{A + B}{2}\cot\dfrac{A - B}{2}.$

21. $\dfrac{\cos A + \cos B}{\cos B - \cos A} = \cot \dfrac{A+B}{2} \cot \dfrac{A-B}{2}$

22. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \dfrac{A+B}{2}.$

23. $\dfrac{\sin A - \sin B}{\cos B - \cos A} = \cot \dfrac{A+B}{2}.$

24. $\dfrac{\cos(A+B+C) + \cos(-A+B+C) + \cos(A-B+C) + \cos(A+B-C)}{\sin(A+B+C) + \sin(-A+B+C) - \sin(A-B+C) + \sin(A+B-C)} = \cot B.$

25. $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4 \cos 4A \cos 5A \cos 6A.$

26. $\cos(-A+B+C) + \cos(A-B+C) + \cos(A+B-C) + \cos(A+B+C) = 4 \cos A \cos B \cos C.$

27. $\sin 50° - \sin 70° + \sin 10° = 0.$

28. $\sin 10° + \sin 20° + \sin 40° + \sin 50° = \sin 70° + \sin 80°.$

29. $\sin a + \sin 2a + \sin 4a + \sin 5a = 4 \cos \dfrac{a}{2} \cos \dfrac{3a}{2} \sin 3a.$

Simplify

30. $\cos\left\{\theta + \left(n - \dfrac{3}{2}\right)\phi\right\} - \cos\left\{\theta + \left(n + \dfrac{3}{2}\right)\phi\right\}.$

31. $\sin\left\{\theta + \left(n + \dfrac{1}{2}\right)\phi\right\} + \sin\left\{\theta + \left(n + \dfrac{1}{2}\right)\phi\right\}.$

97. The formulae (1), (2), (3), and (4) of Art. 94 are also very important. They should be remembered in the form

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B) \quad \ldots(1),$$
$$2 \cos A \sin B = \sin(A+B) - \sin(A-B) \quad \ldots(2),$$
$$2 \cos A \cos B = \cos(A+B) + \cos(A-B) \quad \ldots(3),$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B) \quad \ldots(4).$$

They may be looked upon as the converse of the formulae I–IV of Art. 94.

Ex. 1. $2 \sin 3\theta \cos \theta = \sin 4\theta + \sin 2\theta.$

Ex. 2. $2 \sin 5\theta \sin 3\theta = \cos 2\theta - \cos 8\theta.$

Ex. 3. $2 \cos 11\theta \cos 2\theta = \cos 13\theta + \cos 9\theta.$

Ex. 4. Simplify

$$\frac{\sin 8\theta \cos \theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos \theta - \sin 3\theta \sin 4\theta}$$

By the above formulae, the expression

$$= \frac{\frac{1}{2}[\sin 9\theta + \sin 7\theta] - \frac{1}{2}[\sin 9\theta + \sin 3\theta]}{\frac{1}{2}[\cos 3\theta + \cos \theta] - \frac{1}{2}[\cos \theta - \cos 7\theta]}$$

$$= \frac{\sin 7\theta - \sin 3\theta}{\cos 3\theta + \cos 7\theta}$$

$$= \frac{2\cos 5\theta \sin 2\theta}{2\cos 5\theta \cos 2\theta}, \text{ by the formulae of Art. 94,}$$

$$= \tan 2\theta.$$

[The student should carefully notice the artifice of first employing the formulae of this article and then, to obtain a further simplification, employing the *converse* formulae of Art. 94. This artifice is often successful in simplifications.]

EXAMPLES. XV.

Express as a sum or difference the following :

1. $2\sin 5\theta \sin 7\theta$.
2. $2\cos 7\theta \sin 5\theta$.
3. $2\cos 11\theta \cos 3\theta$.
4. $2\sin 54° \sin 66°$.

Prove that

5. $\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta \sin 5\theta$.
6. $\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$.
7. $\sin A \sin(A + 2B) - \sin B \sin(B + 2A) = \sin(A - B)\sin(A + B)$
8. $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$
9. $\dfrac{2\sin(A - C)\cos C - \sin(A - 2C)}{2\sin(B - C)\cos C - \sin(B - 2C)} = \dfrac{\sin A}{\sin B}$.
10. $\dfrac{\sin A \sin 2A + \sin 3A \sin 6A + \sin 4A \sin 13A}{\sin A \cos 2A + \sin 3A \cos 6A + \sin 4A \cos 13A} = \tan 9A$
11. $\dfrac{\cos 2A \cos 3A - \cos 2A \cos 7A + \cos A \cos 10A}{\sin 4A \sin 3A - \sin 2A \sin 5A + \sin 4A \sin 7A} = \cot 6A \cot 5A$.
12. $\cos(36° - A)\cos(36° + A) + \cos(54° + A)\cos(54° - A) = \cos 2A$.
13. $\cos A \sin(B - C) + \cos B \sin(C - A) + \cos C \sin(A - B) = 0$.
14. $\sin(45° + A)\sin(45° - A) = \frac{1}{2}\cos 2A$.
15. $\text{versin}(A + B)\,\text{versin}(A - B) = (\cos A - \cos B)^2$.
16. $\sin(\beta - \gamma)\cos(\alpha - \delta) + \sin(\gamma - \alpha)\cos(\beta - \delta) + \sin(\alpha - \beta)\cos(\gamma - \delta) = 0$.
17. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$.

98. *To prove that* $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, *and that*

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

By Art. 88, we have, for all values of A and B,

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}}, \text{ by dividing both}$$

numerator and denominator by $\cos A\cos B$.

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Again, by Art. 90,

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$$

$$= \frac{\dfrac{\sin A}{\cos A} - \dfrac{\sin B}{\cos B}}{1 + \dfrac{\sin A\sin B}{\cos A\cos B}}, \text{ by dividing as before.}$$

$$\therefore \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

99. The formulae of the preceding article may be obtained geometrically from the figures of Arts. 88 and 90.

(1) Taking the figure of Art. 88, we have

$$\tan(A+B) = \frac{MP}{OM} = \frac{QN+RP}{OQ-RN}$$

$$= \frac{\dfrac{QN}{OQ} + \dfrac{RP}{OQ}}{1 - \dfrac{RN}{OQ}} = \frac{\tan A + \dfrac{RP}{OQ}}{1 - \dfrac{RN}{RP}\dfrac{RP}{OQ}}.$$

But, since the angles RPN and QON are equal, the triangles RPN and QON are similar, so that

$$\frac{RP}{PN} = \frac{OQ}{ON},$$

and therefore $\dfrac{RP}{OQ} = \dfrac{PN}{ON} = \tan B,$

Hence $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan RPN\tan B} = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}.$

(2) Taking the figure of Art. 90, we have

$$\tan(A-B) = \frac{MP}{OM} = \frac{QN-PR}{OQ+NR}$$

$$= \frac{\dfrac{QN}{OQ} - \dfrac{PR}{OQ}}{1 + \dfrac{NR}{OQ}} = \frac{\tan A - \dfrac{PR}{OQ}}{1 + \dfrac{NR}{PR}\dfrac{PR}{OQ}}.$$

But, since the angles RPN and NOQ are equal, we have $\frac{RP}{PN} = \frac{OQ}{ON}$.

and therefore $\frac{PR}{OQ} = \frac{PN}{ON} = \tan B$.

Hence $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan RPN \tan B} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

100. As particular cases of the preceding formulae, we have, by putting B equal to 45°,

$$\tan(A + 45°) = \frac{\tan A + 1}{1 - \tan A} = \frac{1 + \tan A}{1 - \tan A},$$

and $$\tan(A - 45°) = \frac{\tan A - 1}{1 + \tan A}.$$

Similarly, as in Art. 98, we may prove that

$$\cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$$

and $$\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

101. **Ex. 1.** $\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} + 1)^2}{3 - 1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

$$= 2 + 1{\cdot}73205 \ldots = 3{\cdot}73205 \ldots .$$

Ex. 2. $\tan 45° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)^2}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

$$= 2 - 1{\cdot}73205 \ldots = {\cdot}26795 \ldots$$

EXAMPLES. XVI.

1. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, find the values of $\tan(2A + B)$ and $\tan(2A - B)$. Verify by a graph and accurate measurement.

2. If $\tan A = \frac{\sqrt{3}}{4 - \sqrt{3}}$ and $\tan B = \frac{\sqrt{3}}{4 + \sqrt{3}}$, prove that
$$\tan(A - B) = {\cdot}375.$$

3. If $\tan A = \frac{n}{n + 1}$ and $\tan B = \frac{1}{2n + 1}$, find $\tan(A + B)$.

4. If $\tan\alpha = \frac{5}{6}$ and $\tan\beta = \frac{1}{11}$, prove that $\alpha + \beta = \frac{\pi}{4}$. Verify by a graph and accurate measurement.

Prove that

5. $\tan\left(\frac{\pi}{4} + \theta\right) \times \tan\left(\frac{3\pi}{4} + \theta\right) = -1.$

6. $\cot\left(\frac{\pi}{4} + \theta\right) \times \cot\left(\frac{\pi}{4} - \theta\right) = 1.$

7. $1 + \tan A \tan\frac{A}{2} = \tan A \cot\frac{A}{2} - 1 = \sec A.$

8. Construct the acute angles whose tangents are $\frac{1}{3}$ and $\frac{1}{2}$, and verify by measurement that their sum is 45°.

9. The tangents of two acute angles are respectively 3 and 2; shew by a graph that the tangent of their difference is $\frac{1}{7}$.

10. The sine of one acute is ·6 and the cosine of another is ·5. Shew graphically, and also by calculation, that the sine of their difference is ·39 nearly.

11. Draw the positive angle whose cosine is ·4 and shew, both by measurement and calculation, that the sine and cosine of an angle which exceeds it by 45° and · 93 and − ·365 nearly.

12. Draw the acute angle whose tangent is 7 and the acute angle whose sine is ·7 and shew, both by measurement and calculation, that the sine of their difference is approximately ·61.

102. As further examples of the use of the formulae of the present chapter we shall find the general value or the angle which has a given sine, cosine or tangent. This has been already found in Arts. 82-84.

Find the general value of all angles having a given sine.

Let α be any angle having the given sine and θ any other angle having the same sine.

We have then to find the most general value of θ which satisfies the equation.

$$\sin\theta = \sin\alpha. \text{ i.e. } \sin\theta - \sin\alpha = 0.$$

This may be written

$$2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0,$$

and it is therefore satisfied by

$$\cos\frac{\theta+\alpha}{2} = 0, \text{ and by } \sin\frac{\theta-\alpha}{2} = 0,$$

$$\left.\begin{aligned} \textit{i.e.}\ \text{by } \frac{\theta+\alpha}{2} &= \text{any odd multiple of } \frac{\pi}{2} \\ \text{and by } \frac{\theta-\alpha}{2} &= \text{any multiple of } \pi \end{aligned}\right\},$$

i.e. by $\theta = -\alpha +$ any odd multiple of π ...(1),

and $\theta = \alpha +$ any even multiple of π ...(2),

i.e. θ must $= (-1)^n \alpha + n\pi$, where n is zero or any positive or negative integer.

For, when n is odd, this expression agrees with (1), and, when n is even, it agrees with (2).

103. *Find the general value of all angles having the same cosine.*

The equation we have now to solve is

$$\cos\theta = \cos\alpha,$$

i.e. $$\cos\alpha - \cos\theta = 0,$$

i.e. $$2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0,$$

and it is therefore satisfied by

$$\sin\frac{\theta+\alpha}{2} = 0, \text{ and by } \sin\frac{\theta-\alpha}{2} = 0,$$

i.e. by $\dfrac{\theta+\alpha}{2} =$ any multiple of π,

and by $\dfrac{\theta-\alpha}{2} =$ any multiple of π,

i.e. by $\theta = -\alpha +$ any multiple of 2π,

and by $\theta = \alpha +$ any multiple of 2π.

Both these sets of values are included in the solution $\theta = 2n\pi \pm \alpha$, where n is zero or any positive or negative integer.

104. *Find the general value of all angles having the same tangent.*

The equation we have now to solve is

$$\tan\theta - \tan\alpha = 0,$$

i.e. $$\sin\theta\cos\alpha - \cos\theta\sin\alpha = 0,$$

i.e. $$\sin(\theta - \alpha) = 0.$$

$\therefore$ $\theta - \alpha =$ any multiple of π

$= n\pi$, where n is zero or any positive or negative integer,

so that the most general solution is $\theta = n\pi + \alpha$.

CHAPTER VIII

THE TRIGONMETRICAL RATIOS OF MULTIPLE AND SUBMULTIPLE ANGLES

105. *To find the trigonometricl ratios of an angle 2A in terms of those of the angle A.*

If in the formulae of Art. 88 we put $B = A$, we have

$$\sin 2A = \sin A \cos A + \cos A \sin A = = 2 \sin A \cos A,$$

$$\cos 2A = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$$

$$= (1 - \sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A$$

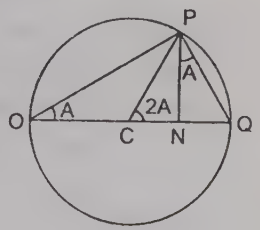
and also $$= \cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1;$$

and $$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \cdot \tan A} = \frac{2 \tan A}{1 - \tan^2 A}.$$

Now the formulae of Art. 88 are true for all values of A and B; hence any formulae derived from them are true for all values of the angles.

In particulars the above formulae are true for all values of A.

106. An independent geometrical proof of the formulae of the preceding article may be given for values of A which are less than a right angle.



Let QCP be the angle $2A$.

With centre C and radius CP describe a circle, and let QC meet it again in O.

Join OP and PQ, and draw PN perpendicular to OQ.

By Geometry, the angle

$$QOP = \frac{1}{2} \angle QCP = A,$$

and the angle $$NPQ = \angle QOP = A.$$

Hence

$$\sin 2A = \frac{NP}{CP} = \frac{2NP}{2CQ} = 2 \frac{NP}{OQ} = 2 \frac{NP}{OP} \cdot \frac{OP}{OQ}$$

$$= 2 \sin NOP \cos POQ, \text{ since } OPQ \text{ is a right angle.}$$

$$= 2 \sin A \cos A;$$

also

$$\cos 2A = \frac{CN}{CP} = \frac{2CN}{OQ} = \frac{(OC + CN) - (OC - CN)}{OQ}$$
$$= \frac{ON - NQ}{OQ} = \frac{ON}{OP}\cdot\frac{OP}{OQ} - \frac{NQ}{PQ}\cdot\frac{PQ}{OQ}$$
$$= \cos^2 A - \sin^2 A\ ;$$

and $$\tan 2A = \frac{NP}{CN} = \frac{2NP}{ON - NQ} = \frac{2\dfrac{NP}{ON}}{1 - \dfrac{NQ}{PN}\cdot\dfrac{PN}{ON}}$$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

Ex. *To find the values of* $\sin 15°$ *and* $\cos 15°$.

Let the angle $2A$ be $30°$, so that A is $15°$.

Let the radius CP be $2a$, so that we have

$$CN = 2a\cos 30° = a\sqrt{3},$$

and $$NP = 2a\sin 30° = a,$$

Hence $$ON = OC + CN = a(2 + \sqrt{3}),$$

and $$NQ = CQ - CN = a(2 - \sqrt{3}).$$

$$\therefore\ OP^2 = ON\,.\,OQ = a(2 + \sqrt{3}) \times 4a,$$

so that $$OP = a\sqrt{2}\,(\sqrt{3} + 1),$$

and $$PQ^2 = QN\cdot QO = a(2 - \sqrt{3}) \times 4a,$$

so that $$PQ = a\sqrt{2}\,(\sqrt{3} - 1).$$

Hence $$\sin 15° = \frac{PQ}{OQ} = \frac{\sqrt{2}\,(\sqrt{3} - 1)}{4} = \frac{\sqrt{3} - 1}{2\sqrt{2}},$$

and $$\cos 15° = \frac{OP}{OQ} = \frac{\sqrt{2}\,(\sqrt{3} + 1)}{4} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

107. *To find the trigonometrical functions of* $3A$ *in terms of those of* A.

By Art, 88, putting B equal to $2A$, we have

$$\sin 3A = \sin(A + 2A) = \sin A\cos 2A + \cos A\sin 2A$$
$$= \sin A(1 - 2\sin^2 A) + \cos A\cdot 2\sin A\cos A,$$

by Art, 105,

$$= \sin A(1 - 2\sin^2 A) + 2\sin A(1 - \sin^2 A).$$

Hence $$\mathbf{\sin 3A = 3\sin A - 4\sin^3 A} \qquad ...(1)$$

So $$\cos 3A = \cos(A + 2A) = \cos A\cos 2A - \sin A\sin 2A$$
$$= \cos A(2\cos^2 A - 1) - \sin A\cdot 2\sin A\cos A$$
$$= \cos A(2\cos^2 A - 1) - 2\cos A(1 - \cos^2 A)$$

Hence $$\mathbf{\cos 3A = 4\cos^3 A - 3\cos A} \qquad ...(2)$$

Also $$\tan 3A = \tan(A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A\tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \cdot \dfrac{2\tan A}{1-\tan^2 A}} = \frac{\tan A\,(1-\tan^2 A) + 2\tan A}{(1-\tan^2 A) - 2\tan^2 A}$$

Hence $$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \qquad \ldots(3)$$

[The student may find it difficult to remember, and distinguish between, the formulae (1) and (2), which bear a general resemblance to one another, but have their signs in a different order. If in doubt, he may always verify his formula by testing it for a particular case, *e.g.* by putting $A = 30°$ for formula (1), and by putting $A = 0°$ for formula (2).]

108. By a process similar to that of the previous article, the trigonometrical ratios of any higher multiples of θ may be expressed in terms of those of θ. The method is however long and tedious. In a later chapter better methods will be pointed out.

As an example, let us express $\cos 5\theta$ in terms of $\cos\theta$.

We have

$$\begin{aligned}
\cos 5\theta &= \cos(3\theta + 2\theta) \\
&= \cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta \\
&= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1) \\
&\qquad - (3\sin\theta - 4\sin^3\theta)\cdot 2\sin\theta\cos\theta \\
&= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) \\
&\qquad - 2\cos\theta\cdot\sin^2\theta\,(3 - 4\sin^2\theta) \\
&= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) \\
&\qquad - 2\cos\theta\,(1-\cos^2\theta)(4\cos^2\theta - 1) \\
&= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta) \\
&\qquad - 2\cos\theta\,(5\cos^2\theta - 4\cos^4\theta - 1) \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}$$

EXAMPLES. XVII.

1. Find the value of $\sin 2a$ when

(1) $\cos a = \frac{3}{5}$, (2) $\sin a = \frac{12}{13}$, and (3) $\tan a = \frac{16}{63}$.

2. Find the value of $\cos 2a$ when

(1) $\cos a = \frac{15}{17}$, (2) $\sin a = \frac{4}{5}$, and (3) $\tan a = \frac{5}{12}$.

Verify by a graph and accurate measurement.

3. If $\tan\theta = \frac{b}{a}$, find the value of $a\cos 2\theta + b\sin 2\theta$.

Prove that

4. $\dfrac{\sin 2A}{1+\cos 2A} = \tan A.$

5. $\dfrac{\sin 2A}{1-\cos 2A} = \cot A.$

6. $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A.$

7. $\tan A + \cot A = 2\operatorname{cosec} 2A.$

8. $\tan A - \cot A = -2\cot 2A.$

9. $\operatorname{cosec} 2A + \cot 2A = \cot A.$

10. $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\dfrac{A}{2}\cot\dfrac{B}{2}.$

11. $\dfrac{\cos A}{1\pm\sin A} = \tan\left(45^\circ \pm \dfrac{A}{2}\right).$

12. $\dfrac{\sec 8A - 1}{\sec 4A - 1} = \dfrac{\tan 8A}{\tan 2A}.$

13. $\dfrac{1+\tan^2(45^\circ - A)}{1-\tan^2(45^\circ - A)} = \operatorname{cosec} 2A.$

14. $\dfrac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta} = \dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}.$

15. $\dfrac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B} = \tan(A+B).$

16. $\tan\left(\dfrac{\pi}{4}+\theta\right) - \tan\left(\dfrac{\pi}{4}-\theta\right) = 2\tan 2\theta.$

17. $\dfrac{\cos A+\sin A}{\cos A-\sin A} - \dfrac{\cos A-\sin A}{\cos A+\sin A} = 2\tan 2A.$

18. $\cot(A+15^\circ) - \tan(A-15^\circ) = \dfrac{4\cos 2A}{1+2\sin 2A}.$

19. $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

20. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}.$

21. $\dfrac{\sin(n+1)A - \sin(n-1)A}{\cos(n+1)A + 2\cos nA + \cos(n-1)A} = \tan\dfrac{A}{2}.$

22. $\dfrac{\sin(n+1)A + 2\sin nA + \sin(n-1)A}{\cos(n-1)A - \cos(n+1)A} = \cot\dfrac{A}{2}.$

23. $\sin(2n+1)A\sin A = \sin^2(n+1)A - \sin^2 nA.$

24. $\dfrac{\sin(A+3B)+\sin(3A+B)}{\sin 2A+\sin 2B} = 2\cos(A+B).$

25. $\sin 3A + \sin 2A - \sin A = 4\sin A\cos\dfrac{A}{2}\cos\dfrac{3A}{2}.$

26. $\tan 2A = (\sec 2A + 1)\sqrt{\sec^2 A - 1}.$

27. $\cos^3 2\theta + 3\cos 2\theta = 4(\cos^6\theta - \sin^6\theta).$

28. $1+\cos^2 2\theta = 2(\cos^4\theta + \sin^4\theta).$

29. $\sec^2 A\ (1 + \sec 2A) = 2 \sec 2A$.

30. $\operatorname{cosec} A - 2 \cot 2A \cos A = 2 \sin A$.

Prove that

31. $\cot A = \frac{1}{2}\left(\cot\frac{A}{2} - \tan\frac{A}{2}\right)$.

32. $\sin a \sin (60° - a) \sin (60° + a) = \frac{1}{4} \sin 3a$.

33. $\cos a \sin (60° - a) \cos (60° + a) = \frac{1}{4} \cos 3a$.

34. $\cot a + \cos (60° + a) - \cot (60° - a) = 3 \cot 3a$.

35. $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$.

36. $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$.

37. $\cos 4a = 1 - 8 \cos^2 a + 8 \cos^4 a$.

38. $\sin 4A = 4 \sin A \cos^3 A - 4 \cos A \sin^3 A$.

39. $\cos 6a = 32 \cos^6 a - 48 \cos^4 a + 18 \cos^2 a - 1$.

40. $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$.

41. $$\frac{2 \cos 2^n \theta + 1}{2 \cos \theta + 1} = (2 \cos\theta - 1)(2 \cos 2\theta - 1)(2 \cos 2^2 \theta - 1)$$
$$\ldots\ldots(2 \cos 2^{n-1} \theta - 1)$$

Submultiple angles.

109. Since the relations of Art. 105 are true for *all* values of the angle A, they will be true is instead of A we substitute $\frac{A}{2}$, and therefore if instead of $2A$ we put

2. $\frac{A}{2}$, *i.e* A.

Hence we have the relations

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \qquad \ldots(1),$$

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

$$= 2 \cos^2 \frac{A}{2} - 1 = 1 - 2 \sin^2 \frac{A}{2} \qquad \ldots(2),$$

and $$\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} \qquad \ldots(3),$$

From (1), we also have

$$\sin A = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2}}$$

$$= \frac{2\tan\frac{A}{2}}{1 + \tan^2\frac{A}{2}},$$

by dividing both numerator and denominator by $\cos^2\frac{A}{2}$.

So, $$\cos A = \frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2}}$$

$$= \frac{1 - \tan^2\frac{A}{2}}{1 + \tan^2\frac{A}{2}}.$$

110. *To express the trigonometrical ratios of the angle* $\frac{A}{2}$ *is terms of* $\cos A$.

From equation (2) of the previous article, we have

$$\cos A = 1 - 2\sin^2\frac{A}{2},$$

so that $$2\sin^2\frac{A}{2} = 1 - \cos A,$$

and therefore $$\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}} \qquad ...(1)$$

Again, $$\cos A = 2\cos^2\frac{A}{2} - 1$$

so that $$2\cos^2\frac{A}{2} = 1 + \cos A,$$

and therefore $$\cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$$

Hence, $$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}} \qquad ...(3)$$

111. In each of the preceding formulae it will be noted that there is an ambiguous sing. In any particular case the proper sign can be determined as the following examples will shew.

Ex. 1. *Given* $\cos 45° = \frac{1}{\sqrt{2}}$, *find the values of* $\sin 22\frac{1}{2}°$ *and* $\cos 22\frac{1}{2}°$.

The equation (1) of the previous article gives, by putting A equal to 45°,

$$\sin 22\tfrac{1}{2}° = \pm\sqrt{\frac{1-\cos 45°}{2}} = \pm\sqrt{\frac{1-1/\sqrt{2}}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}}$$
$$= \pm\frac{1}{2}\sqrt{2-\sqrt{2}}.$$

Now $\sin 22\frac{1}{2}°$ is necessarily positive, so that the upper sign must be taken.

Hence $\sin 22\frac{1}{2}° = \frac{1}{2}\sqrt{2-\sqrt{2}}$

So, $\cos 22\frac{1}{2}° = \pm\sqrt{\frac{1+\cos 45°}{2}} = \pm\sqrt{\frac{2+\sqrt{2}}{4}} = \pm\frac{1}{2}\sqrt{2+\sqrt{2}}$;

also $\cos 22\frac{1}{2}°$ is positive;

$$\cos 22\tfrac{1}{2}° = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Ex. 2. *Given* $\cos 330° = \frac{\sqrt{3}}{2}$, *find the values of* $\sin 65°$ *and* $\cos 165°$.

The equation (1) gives

$$\sin 165° = \pm\sqrt{\frac{1-\cos 330°}{2}} = \pm\sqrt{\frac{1-\sqrt{3}/2}{2}} = \pm\sqrt{\frac{4-2\sqrt{3}}{8}}$$
$$= \pm\frac{\sqrt{3}-1}{2\sqrt{2}}$$

Also

$$\cos 165° = \pm\sqrt{\frac{1+\cos 330°}{2}} = \pm\sqrt{\frac{1+\sqrt{3}/2}{2}} = \pm\sqrt{\frac{4+2\sqrt{3}}{8}}$$
$$= \pm\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Now 165° lies between 90° and 180°, so that, by Art, 52, its sine is positive and its cosine is negative.

Hence $\sin 165° = \frac{\sqrt{3}-1}{2\sqrt{2}}$,

and $\cos 165° = -\frac{\sqrt{3}+1}{2\sqrt{2}}$.

From the above examples it will be seen that, when the angle A and its cosine are given, the ratios for the angle $\frac{A}{2}$ may be determined without any ambiguity of sign.

When, however, only $\cos A$ is given, there is an ambiguity in finding $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$. The explanation of this ambiguity is given in the next article.

**112. *To explain why there is ambiguity when* $\cos\frac{A}{2}$ *and* $\sin\frac{A}{2}$ *are found from the value of* $\cos A$.

We know that, if n be any integer,

$$\cos A = \cos(2n\pi \pm A) = k \text{ (say)}.$$

Hence any formula which gives us $\cos\frac{A}{2}$ in terms of k should give us also the cosine of $\frac{2n\pi \pm A}{2}$.

Now $$\cos\frac{2n\pi \pm A}{2} = \cos\left(n\pi \pm \frac{A}{2}\right)$$

$$= \cos n\pi \cos\frac{A}{2} \mp \sin n\pi \sin\frac{A}{2} = \cos n\pi \cos\frac{A}{2}$$

$$= \pm\cos\frac{A}{2},$$

according as n is even or odd.

Similarly, any formula, giving us $\sin\frac{A}{2}$ in terms of k, should give us also the sine of $\frac{2n\pi \pm A}{2}$.

Also $$\sin\frac{2n\pi \pm A}{2} = \sin\left(n\pi \pm \frac{A}{2}\right)$$

$$= \sin n\pi \cos\frac{A}{2} \pm \cos n\pi \sin\frac{A}{2}$$

$$= \pm\cos n\pi \sin\frac{A}{2}$$

$$= \pm\sin\frac{A}{2}.$$

Hence, in each case, we should expect to obtain two values for $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$, and this is the number which the formulae of Art. 110 give.

[The student may illustrate this article geometrically by drawing the angles $\frac{2n\pi \pm A}{2}$, i.e., $n\pi \pm \frac{A}{2}$. The bounding line for these angles will have four positions, two inclined to the positive direction of the initial line at angles $\frac{A}{2}$ and $-\frac{A}{2}$, and two inclined at $\frac{A}{2}$ and $-\frac{A}{2}$ to the negative direction of the initial line. It will be clear from the figure that there are two values for $\cos\frac{A}{2}$ and two for in $\sin\frac{A}{2}$.]

113. *To express the trigonometrical ratios of the angle* $\frac{A}{2}$ *in terms of* $\sin A$.

From equation (1) of Art. 109, we have

$$2 \sin \frac{A}{2} \cos \frac{A}{2} = \sin A \qquad \ldots(1)$$

Also $$\sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} = 1, \text{ always} \qquad \ldots(2)$$

First adding these equations, and then subtracting (1) from (2), we have

$$\sin^2 \frac{A}{2} + 2 \sin \frac{A}{2} \cos \frac{A}{2} + \cos^2 \frac{A}{2} = 1 + \sin A,$$

and $$\sin^2 \frac{A}{2} - 2 \sin \frac{A}{2} \cos \frac{A}{2} + \cos^2 \frac{A}{2} = 1 - \sin A;$$

i.e. $$\left(\sin \frac{A}{2} + \cos \frac{A}{2}\right)^2 = 1 + \sin A,$$

and $$\left(\sin \frac{A}{2} - \cos \frac{A}{2}\right)^2 = 1 - \sin A;$$

so that $$\sin \frac{A}{2} + \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} \qquad \ldots(3)$$

and $$\sin \frac{A}{2} - \cos \frac{A}{2} = \pm \sqrt{1 - \sin A} \qquad \ldots(4)$$

By adding, and then subtracting, we have

$$2 \sin \frac{A}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A} \qquad \ldots(5)$$

and $$2 \cos \frac{A}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A} \qquad \ldots(6)$$

The other ratios of $\frac{A}{2}$ are then easily obtained.

114. In each of the formulae (5) and (6) there are two ambiguous signs. In the following examples it is shewn how to determine the ambiguity in any particular case.

Ex. 1. *Given that* $\sin 30°$ *is* $\frac{1}{2}$, *find the values of* $\sin 15°$ *and* $\cos 15°$.

Putting $A = 30°$ we have from relations (3) and (4),

$$\sin 15° + \cos 15° = \pm \sqrt{1 + \sin 30°} = \pm \frac{\sqrt{3}}{\sqrt{2}},$$

$$\sin 15° - \cos 15° = \pm \sqrt{1 - \sin 30°} = \pm \frac{1}{\sqrt{2}}.$$

Now $\sin 15°$ and $\cos 15°$ are both positive, and $\cos 15°$ is greater than $\sin 15°$. Hence the expressions $\sin 15° + \cos 15°$ and $\sin 15° - \cos 15°$ are respectively positive and negative.

Hence the above two relations should be

$$\sin 15° + \cos 15° = + \frac{\sqrt{3}}{\sqrt{2}},$$

and $$\sin 15° - \cos 15° = - \frac{1}{\sqrt{2}}$$

Hence $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$, and $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$.

Ex. 2. *Given that the* 570° *is equal to* $-\frac{1}{2}$, *find the values of* sin 285° *and* cos 285°.

Putting A equal to 570°, we have

$$\sin 285° + \cos 285° = \pm\sqrt{1 + \sin 570°} = \pm\frac{1}{\sqrt{2}},$$

and $$\sin 285° - \cos 285° = \pm\sqrt{1 - \sin 570°} = \pm\sqrt{\frac{3}{2}}.$$

Now sin 285° is negative, cos 285° is positive, and the former is numerically greater than the latter, as may be seen by a figure.

Hence sin 285° + cos 285° is negative, and sin 285° − cos 285° is also negative.

$$\therefore \quad \sin 285° + \cos 285° = - \frac{1}{\sqrt{2}},$$

and $$\sin 285° - \cos 285° = - \frac{\sqrt{3}}{\sqrt{2}},$$

Hence $$\sin 285° = - \frac{\sqrt{3}+1}{2\sqrt{2}},$$

and $$\cos 285° = - \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

**115. *To explain why there is ambiguity when* $\sin\frac{A}{2}$ *and* $\cos\frac{A}{2}$ *are found from the value of* sin A.

We know that, if n be any integer.

$$\sin\{n\pi + (-1)^n A\} = \sin A = k \text{ (say).} \qquad \text{(Art. 82)}$$

Hence any formula which gives us $\sin\frac{A}{2}$ in terms of k should give us also the sine of $\frac{n\pi + (-1)^n A}{2}$.

First, let n be even and equal to $2m$. Then

$$\sin\frac{n\pi + (-1)^n A}{2} = \sin\left(m\pi + \frac{A}{2}\right)$$

$$= \sin m\pi \cos\frac{A}{2} + \cos m\pi \sin\frac{A}{2} = \cos m\pi \sin\frac{A}{2}$$

$$= \pm \sin\frac{A}{2},$$

according as m is even or odd.

Secondly, let n be odd and equal to $2p + 1$,

Then

$$\sin\frac{n\pi + (-1)^n A}{2} = \sin\frac{2p\pi + \pi - A}{2} = \sin\left[p\pi + \frac{\pi - A}{2}\right]$$

$$= \sin p\pi \cos\frac{\pi - A}{2} + \cos p\pi \sin\frac{\pi - A}{2} = \cos p\pi \cos\frac{A}{2}$$

$$= \pm\cos\frac{A}{2},$$

according as p is even or odd.

Hence any formula which gives us $\sin\frac{A}{2}$ in terms of $\sin A$ should be expected to give us, in addition, the values of

$$-\sin\frac{A}{2}, \cos\frac{A}{2} \text{ and } -\cos\frac{A}{2},$$

i.e. four values in all. This is the number of values which we get from the formulae of Art. 113, by giving all possible values to the ambiguities.

In a similar manner it may be shewn that when $\cos\frac{A}{2}$ is found from $\sin A$, we should expect four values.

[If the angle $\frac{n\pi + (-1)^n A}{2}$, i.e. $n\frac{\pi}{2} + (-1)^n \frac{A}{2}$, be drawn geometrically for the case when $\frac{A}{2}$ is an acute angle, it will be found that there are four positions of the bounding line, two in the first quadrant inclined at angles $\frac{A}{2}$ and $\frac{\pi}{2} - \frac{A}{2}$ to the initial line, and two in the third quadrant inclined at $\frac{A}{2}$ and $\frac{\pi}{2} - \frac{A}{2}$ to the negative direction of the initial line. it will be clear from the figure that we should then expect four values for $\sin\frac{A}{2}$ and four for $\cos\frac{A}{2}$. Similarly for any other values of $\frac{A}{2}$.]

116. *In any general case we can shew how the ambiguities in relations (3) and (4) of Art. 113 may be found.*

We have

$$\sin\frac{A}{2} + \cos\frac{A}{2} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\frac{A}{2} + \frac{1}{\sqrt{2}}\cos\frac{A}{2}\right)$$

$$= \sqrt{2}\left[\sin\frac{A}{2}\cos\frac{\pi}{4} + \cos\frac{A}{2}\sin\frac{\pi}{4}\right] = \sqrt{2}\sin\left(\frac{\pi}{4} + \frac{A}{2}\right).$$

The right-hand member of this equation is positive if

$$\frac{\pi}{4} + \frac{A}{2} \text{ lie between } 2n\pi \text{ and } 2n\pi + \pi,$$

i.e. if $\frac{A}{2}$ lie between $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$.

Hence $\sin\frac{A}{2} + \cos\frac{A}{2}$ is positive if

$\frac{A}{2}$ lie between $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$;

it is negative otherwise.

Similarly we can prove that

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \sqrt{2}\sin\left(\frac{A}{2} - \frac{\pi}{4}\right).$$

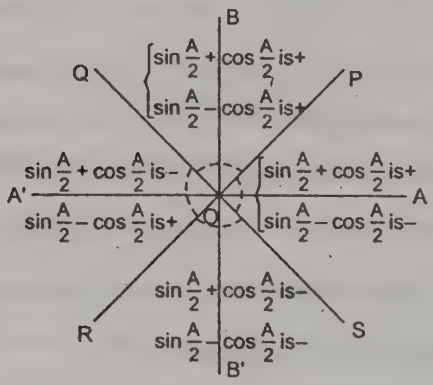
Therefore $\sin\frac{A}{2} - \cos\frac{A}{2}$ is positive if

$\left(\frac{A}{2} - \frac{\pi}{4}\right)$ lie between $2n\pi$ and $2n\pi + \pi$,

i.e. if $\frac{A}{2}$ lie between $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$.

It is negative otherwise,

The results of this article are shewn graphically in the following figure.



OA is the initial line, and *OP*, *OQ*, *OR* and *OS* bisect the angles in the first, second, third and fourth quadrants respectively.

Numerical Example. *Within what limits must* $\frac{A}{2}$ *lie if*

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}?$$

In this case the formulae of Art. 113 must clearly be

$$\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A} \qquad \ldots(1)$$

and $$\sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1-\sin A} \qquad ...(2)$$

For the addition of these two formulae gives the given formula.

From (1) it follows that the revolving line which bounds the angle $\frac{A}{2}$ must lie between OQ and OR or else between OR and OS.

From (2) it follows that the revolving line must lie between OR and QS or else between OS and OP.

Both these conditions are satisfied only when the revolving line lies between OR and OS, and therefore the angle $\frac{A}{2}$ lies between.

$$2n\pi - \frac{3\pi}{4} \text{ and } 2n\pi - \frac{\pi}{4}.$$

117. *To express the trigonometrical ratios of* $\frac{A}{2}$ *in terms of* $\tan A$.

From equations (3) or Art. 109, we have

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}.$$

$$\therefore \quad 1-\tan^2\frac{A}{2} = \frac{2}{\tan A}\cdot\tan\frac{A}{2}.$$

Hence $$\tan^2\frac{A}{2} + \frac{2}{\tan A}\cdot\tan\frac{A}{2} + \frac{1}{\tan^2 A} = 1 + \frac{1}{\tan^2 A}$$

$$= \frac{1+\tan^2 A}{\tan^2 A}.$$

$$\therefore \quad \tan\frac{A}{2} + \frac{1}{\tan A} = \pm\frac{\sqrt{1+\tan^2 A}}{\tan A}$$

$$\therefore \quad \tan\frac{A}{2} = \frac{\pm\sqrt{1+\tan^2 A}-1}{\tan A} \qquad ...(1)$$

118. *The ambiguous sign in equation (1) of the previous article can only be determined when we know something of the magnitude of A.*

Ex. Given $\tan 15° = 2-\sqrt{3}$, find $\tan 7\frac{1}{2}°$.

Putting $A = 15°$ we have, from equation (1) of the previous article,

$$\tan 7\frac{1}{2}° = \frac{\pm\sqrt{1+(2-\sqrt{3})^2}-1}{2-\sqrt{3}} = \frac{\pm\sqrt{8}-4\sqrt{3}-1}{2-\sqrt{3}} \qquad ...(1)$$

Now $\tan 7\frac{1}{2}°$ is positive, so that we must take the upper sign.

Hence $\tan 7\frac{1}{2}^\circ = \dfrac{+(\sqrt{6}-\sqrt{2})-1}{2-\sqrt{3}}$

$= (\sqrt{6}-\sqrt{2}-1)(2+\sqrt{3}) = \sqrt{6}-\sqrt{3}+\sqrt{2}-2$

$= (\sqrt{3}-\sqrt{2})(\sqrt{2}-1).$

Since $\tan 15^\circ = \tan 195^\circ$, the equation which gives us $\tan\dfrac{15^\circ}{2}$ in terms of $\tan 15^\circ$ may be expected to give us $\tan\dfrac{195^\circ}{2}$ in terms of $\tan 195^\circ$. In fact the value obtained from (1) by taking the negative sign before the radical is $\tan\dfrac{195^\circ}{2}$.

Hence $\tan\dfrac{195^\circ}{2} = \dfrac{-\sqrt{8-4\sqrt{3}}-1}{2-\sqrt{3}} = \dfrac{-(\sqrt{6}-\sqrt{2})-1}{2-\sqrt{3}}$

$= (-\sqrt{6}+\sqrt{2}-1)(2+\sqrt{3})$

$= -(\sqrt{3}+\sqrt{2})(\sqrt{2}+1).$

that $-\cot 7\frac{1}{2}^\circ = \tan 97\frac{1}{2}^\circ = -(\sqrt{3}+\sqrt{2})(\sqrt{2}+1).$

**119. *To explain why there is ambiguity when* $\tan\dfrac{A}{2}$ *is found from the value of* $\tan A$.

We know, by Art. 84, that if n be any integer,

$$\tan(n\pi + A) = \tan A = k \text{ (say)}$$

Hence any equation which gives us $\tan\dfrac{A}{2}$ in terms of k may be expected to given us $\tan\dfrac{n\pi + A}{2}$ also.

First, let n be even and equal to $2m$.

Then

$$\tan\frac{n\pi + A}{2} = \tan\frac{2m\pi + A}{2} = \tan\left(m\pi + \frac{A}{2}\right)$$

$$= \tan\frac{A}{2}, \text{ as in Art. 84}$$

Secondly, let n be odd and equal to $2p + 1$.

Then $\tan\dfrac{n\pi + A}{2} = \tan\dfrac{(2p+1)\pi + A}{2}$

$$= \tan\left(p\pi + \frac{\pi + A}{2}\right) = \tan\frac{\pi + A}{2} \text{ (Art. 84)}$$

$$= -\cot\frac{A}{2}. \text{ (Art. 70.)}$$

Hence the formula which gives us the value of $\tan\dfrac{A}{2}$ should be expected to give us also the value of $-\cot\dfrac{A}{2}$.

An illustration of this is seen in the example of the previous article.

EXAMPLES XVIII.

1. If $\sin\theta = \frac{1}{2}$ and $\sin\phi = \frac{1}{3}$, find the values of
$$\sin(\theta + \phi) \text{ and } \sin(2\theta + 2\phi).$$
2. The tangent of an angle is 2.4. Find its cosecant, the cosecant of half the angle, and the cosecant of the supplement of double the angle.
3. If $\cos\alpha = \frac{11}{61}$ and $\sin\beta = \frac{4}{5}$, find the values of $\sin^2\frac{\alpha-\beta}{2}$ and $\cos^2\frac{\alpha+\beta}{2}$, the angles α and β being positive acute angles.
4. If $\cos\alpha = \frac{3}{5}$ and $\cos\beta = \frac{4}{5}$, find the value of $\cos\frac{\alpha-\beta}{2}$, the angles α and β being positive acute angles.
5. Given $\sec\theta = 1\frac{1}{4}$, find $\tan\frac{\theta}{2}$ and $\tan\theta$, Verify by a graph.
6. If $\cos A = \cdot 28$ find the value of $\tan\frac{A}{2}$, and explain the resulting ambiguity.
7. Find the values of (1) $\sin 7\frac{1^\circ}{2}$, (2) $\cos 7\frac{1^\circ}{2}$, (3) $\tan 22\frac{1^\circ}{2}$, and (4) $\tan 11\frac{1^\circ}{4}$.
8. If $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$, find the value of $\tan\frac{\theta-\phi}{2}$.

Prove that

9. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2\frac{\alpha+\beta}{2}$.
10. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\frac{\alpha-\beta}{2}$.
11. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$.
12. $\sin A = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$.
13. $\cos A = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$.
14. $\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) = 2\sec 2\theta$.
15. $\tan\left(45^\circ + \frac{A}{2}\right) = \sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sec A + \tan A$
16. $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$.

17. $\cos^2 a + \cos^2 (a + 120°) + \cos^2 (a - 120°) = \frac{3}{2}$.

18. $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{3}{2}$.

19. $\sin^4 \frac{\pi}{8} + \sin^4 \frac{3\pi}{8} + \sin^4 \frac{5\pi}{8} + \sin^4 \frac{7\pi}{8} = \frac{3}{2}$

20. $\cos 2\theta \cos 2\phi + \sin^2 (\theta - \phi) - \sin^2 (\theta + \phi) = \cos (2\theta + 2\phi)$.

21. $(\tan 4A + \tan 2A)(1 - \tan^2 3A \tan^2 A) = 2 \tan 3A \sec^2 A$.

22. $\left(1 + \tan \frac{a}{2} - \sec \frac{a}{2}\right)\left(1 + \tan \frac{a}{2} + \sec \frac{a}{2}\right) + \sin a \sec^2 \frac{a}{2}$

Find the proper signs to be applied to the radicals in the three following formulae :

23. $2 \cos \frac{A}{2} = \pm \sqrt{1 - \sin A} \pm \sqrt{1 + \sin A}$, when $\frac{A}{2} = 278°$.

24. $2 \sin \frac{A}{2} = \pm \sqrt{1 - \sin A} \pm \sqrt{1 + \sin A}$, when $\frac{A}{2} = \frac{19\pi}{11}$.

25. $2 \cos \frac{A}{2} = \pm \sqrt{1 - \sin A} \pm \sqrt{1 + \sin A}$, when $\frac{A}{2} = -140°$.

26. If $A = 340°$, prove that

$$2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$$

and

$$2 \cos \frac{A}{2} = -\sqrt{1 + \sin A} - \sqrt{1 - \sin A}$$

27. If $A = 460°$, prove that

$$2 \cos \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$$

28. If $A = 580°$, prove that

$$2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$$

29. Within what respective limits must $\frac{A}{2}$ lie when

(1) $2 \sin \frac{A}{2} = \sqrt{1 + \sin A} + \sqrt{1 - \sin A}$

(2) $2 \sin \frac{A}{2} = -\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$

(3) $2 \sin \frac{A}{2} = +\sqrt{1 + \sin A} + \sqrt{1 - \sin A}$

and (4) $2 \cos \frac{A}{2} = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}$

30. In the formula

$$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}$$

find within what limits $\frac{A}{2}$ must lie when

(1) the two positive signs are taken,
(2) the two negaive signs are taken,
and (3) the first sign is negative and the second positive.

31. Prove that the sine is algebraically less than the cosine for any angle between $2n\pi - \frac{3\pi}{4}$ and $2n\pi + \frac{\pi}{4}$, where n is any integer.

32. If $\sin\frac{A}{3}$ be determined from the equation

$$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3},$$

prove that we should expect to obtain also the values of

$$\sin\frac{\pi - A}{3} \text{ and } -\sin\frac{\pi + A}{3},$$

Give also a geometrical illustration.

33. If $\cos\frac{A}{3}$ be found from the equation.

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3},$$

prove that we should expect to obtain also the values of

$$\cos\frac{2\pi - A}{3} \text{ and } \cos\frac{2\pi + A}{3},$$

Give also a geometrical illustration.

120. By the use of the formulae of the present chapter we can now find the trigonometrical ratios of some important angles.

To find the trigonometrical functions of an angle of 18°.

Let θ stand for 18°, so that 2θ is 36° and 3θ is 54°.

Hence $2\theta = 90° - 3\theta,$

and therefore

$$\sin 2\theta = \sin(90° - 3\theta) = \cos 3\theta.$$

$\therefore$ $2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$ (Arts. 105 and 107).

Hence, either $\cos\theta = 0$, which gives $\theta = 90°$, or

$$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta.$$

$\therefore$ $4\sin^2\theta + 2\sin\theta = 1$

By solving this quadratic equation, we have

$$\sin\theta = \frac{\pm\sqrt{5} - 1}{4}.$$

In our case $\sin\theta$ is necessarily a positive quantity. Hence we take the upper sign, and have

$$\sin 18° = \frac{\sqrt{5}-1}{4}.$$

Hence

$$\cos = 18° = \sqrt{1-\sin^2 18°} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}}$$

$$= \frac{\sqrt{10+2\sqrt{5}}}{4}$$

The remaining trigonometrical ratios os 18° may be now found.

Since 72° is the complement of 18°, the values of the ratios for 72° may be obtained by the use of Art. 69.

121. *To find the trigonometrical functions of an angle of* 36°.

Since $\cos 2\theta = 1 - 2\sin^2\theta$, (Art. 105),

$$\therefore \quad \cos 36° = 1 - 2\sin^2 18° = 1 - 2\left(\frac{6-2\sqrt{5}}{16}\right)$$

$$= 1 - \frac{3-\sqrt{5}}{4}.$$

so that $$\cos 36° = \frac{\sqrt{5}+1}{4}.$$

Hence

$$\sin 36° = \sqrt{1-\cos^2 36°} = \sqrt{1-\frac{6+2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

The remaining trigonometrical functions of 36° may now be found.

Also, since 54° is the complement of 36°, the values of the functions for 54° may be found by the help of Art. 69.

122. *The value of* $\sin 18°$ *and* $\cos 36°$ *may also be found geometrically as follows.*

Let ABC be a triangle constructed so that each of the angles B and C is double of the angle A. Then

$$180° = A + B + C = A + 2A + 2A,$$

so that $A = 36°$.

Hence, if AD be drawn perpendicular to BC, we have

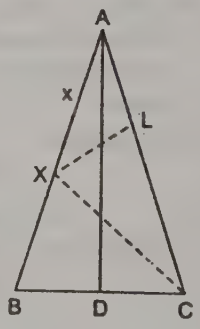
$$\angle BAD = 18°.$$

By the geometrical construction we know that BC is equal to AX where X is a point on AB, such that

$$AB \cdot BX = AX^2$$

Let $AB = a$, and $AX = x$.

This relation then gives



$$a(a-x) = x^2,$$

i.e. $$x^2 + ax = a^2,$$

i.e. $$x = a\frac{\sqrt{5}-1}{2}.$$

Hence $$\sin 18° = \sin BAD = \frac{BD}{BA} = \frac{1}{2}\frac{BC}{BA}$$

$$= \frac{1}{2}\frac{x}{a} = \frac{\sqrt{5}-1}{4}$$

Again, by Geometry, we know that AX and XC are equal; hence, if XL be perpendicular to AC, then L bisects AC.

Hence

$$\cos 36° = \frac{AL}{AX} = \frac{a}{2} \div x = \frac{1}{\sqrt{5}-1}$$

$$= \frac{\sqrt{5}+1}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{\sqrt{5}+1}{4}.$$

123. *To find the trigonometrical functions for an angle of* 9°

Since sin 9° and cos 9° are both positive, the relation (3) of Art. 113 gives

$$\sin 9° + \cos 9° = \sqrt{1 + \sin 18°} = \sqrt{1 + \frac{\sqrt{5}-1}{4}} = \frac{\sqrt{3+\sqrt{5}}}{2} \quad ...(1)$$

Also, since cos 9° is greater than sin 9° (Art. 53), the quantity sin 9° − cos 9° in negative. Hence the relation (4) of Art. 113 gives.

$$\sin 9° - \cos 9° = -\sqrt{1 - \sin 18°} = -\sqrt{1 - \frac{\sqrt{5}-1}{4}}$$

$$= -\frac{\sqrt{5-\sqrt{5}}}{2} \quad ...(2)$$

By adding (1) and (2), we have

$$\sin 9° = \frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4},$$

and, by subtracting (2) from (1), we have

$$\cos 9° = \frac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4}.$$

The remaining functions for 9° may be found.

Also, since 81° is the complement of 9°, the values of the functions for 81° may be obtained by the use of Art. 69.

EXAMPLES. XIX.

Prove that :

1. $\sin^2 72° - \sin^2 60° = \frac{\sqrt{5}-1}{8}.$

2. $\cos^2 48° - \sin^2 12° = \frac{\sqrt{5}+1}{8}.$

3. $\cos 12° + \cos 60° + \cos 84° = \cos 24° + \cos 48°$. Verify by a graph.

4. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$.

5. $\sin\frac{\pi}{10} + \sin\frac{13\pi}{10} = -\frac{1}{2}$.

6. $\sin\frac{\pi}{10}\sin\frac{13\pi}{10} = -\frac{1}{4}$.

7. $\tan 6° \tan 42° \tan 66° \tan 78° = 1$.

8. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2^7}$.

9. $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = 1$.

10. Two parallel chords of a circle, which are on the same side of the centre, subtend angles of 72° and 144° respectively at the centre. Prove that the perpendicular distance between the chords is half the radius of the circle.

11. In any circle prove that the chord which subtends 180° at the centre is equal to the sum of the two chords which subtend angles of 36° and 60°.

12. Construct the angle whose cosine is equal to its tangent.

13. Solve the equation

$$\sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta.$$

CHAPTER IX

IDENTITIES AND TRIGONOMETRICAL EQUATIONS

124. The formulae of Arts. 88 and 90 can be used to obtain the trigonometrical ratios of the sum of more than two angles.

For example

$$\begin{aligned}\sin(A+B+C) &= \sin(A+B)\cos C + \cos(A+B)\sin C \\ &= [\sin A\cos B + \cos A\sin B]\cos C \\ &\qquad + [\cos A\cos B - \sin A\sin B]\sin C \\ &= \sin A\cos B\cos C + \cos A\sin B\cos C \\ &\qquad + \cos A\cos B\sin C - \sin A\sin B\sin C\end{aligned}$$

So

$$\begin{aligned}\cos(A+B+C) &= \cos(A+B)\cos C - \sin(A+B)\sin C \\ &= (\cos A\cos B - \sin A\sin B)\cos C \\ &\qquad - (\sin A\cos B + \cos A\sin B)\sin C \\ &= \cos A\cos B\cos C - \cos A\sin B\sin C \\ &\qquad - \sin A\cos A\sin C - \sin A\sin B\sin C\end{aligned}$$

Also $$\tan(A+B+C) = \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

$$= \frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\tan B}\tan C}$$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}$$

125. The last formula of the previous article is a particular case of a very general theorem which gives the tangent of the sum of any number of angles in terms of the tangents of the angles themselves. The theorem is

$$\tan(A_1+A_2+A_3+\ldots+A_n) = \frac{s_1-s_3+s_5-s_7+\ldots}{1-s_2+s_4-s_6+\ldots} \qquad \ldots(1)$$

where

$s_1 = \tan A_1 + \tan A_2 + \ldots + \tan A_n$

= the sum of the tangents of the separate angles,

$s_2 = \tan A_1\tan A_2 + \tan A_1\tan A_3 + \ldots$

= the sum of the tangents taken two at a time,

$s_3 = \tan A_1 \tan A_2 \tan A_3 = \tan A_2 \tan A_3 \tan A_4 + \ldots$

= the sum of the tangents taken three at a time, and so on,

Assume the relation (1) to hold for n angles, and add on another angle A_{n+1}.

Then $\tan (A_1 + A_2 + \ldots + A_{n+1})$

$$= \tan [(A_1 + A_2 + \ldots + A_n) + A_{n+1}]$$

$$= \frac{\tan (A_1 + A_2 + \ldots + A_n) + \tan A_{n+1}}{1 - \tan (A_1 + A_2 + \ldots + A_n) \cdot \tan A_{n+1}}$$

$$= \frac{\dfrac{s_1 - s_3 + s_5 - s_7 + \ldots}{1 - s_2 + s_4 - \ldots} + \tan A_{n+1}}{1 - \dfrac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - \ldots} \tan A_{n+1}}$$

Let $\tan A_1, \tan A_2, \ldots \tan_{n+1}$ be respectively called $t_1, t_2, \ldots t_{n+1}$.

Then $\tan (A_1 + A_2 + \ldots + A_{n+1})$

$$= \frac{(s_1 - s_3 + s_5 - \ldots) + t_{n+1}(1 - s_2 + s_4 - \ldots)}{1 - s_2 + s_4 - \ldots) - (s_1 - s_3 + s_5 - \ldots) t_{n+1}}$$

$$= \frac{(s_1 + t_{n+1}) - (s_3 + s_2 t_{n+1}) + (s_5 + s_4 t_{n+1}) - \ldots}{1 - (s_2 + s_1 t_{n+1}) + (s_4 + s_3 t_{n+1}) - (s_6 + s_5 t_{n+1}) + \ldots}$$

But $s_1 + t_{n+1} = (t_1 + t_2 + \ldots + t_n) + t_{n+1}$

= the sum of the $(n+1)$ tangents,

$s_2 + s_1 t_{n+1} = (t_1 t_2 + t_2 t_3 + \ldots) + (t_1 + t_2 + \ldots + t_n) t_{n+1}$

= the sum, two at a time, of the $(n+1)$ tangents.

$s_3 + s_2 t_{n+1} = (t_1 t_2 t_3 + t_2 t_3 t_4 + \ldots) + (t_1 t_2 + t_2 t_3 + \ldots) t_{n+1}$

= the sum, three at a time, of the $(n+1)$ tangents, and so on.

Hence we see that the same rule holds for $(n+1)$ angles as for n angles.

Hence, if the theorem be true for n angles, it is true for $(n+1)$ angles.

But, by Arts. 98 and 124, it is true for two and three angles.

Hence, the theorem is true for four angles; hence for five angles ... Hence it is true universally.

Cor. If the angles be all equal, and there be n of them, and each equal to θ, then

$$s_1 = n \tan \theta; s_2 = {}^nC_2 \tan^2 \theta; s_3 = {}^nC_3 \tan^3 \theta, \ldots$$

Ex. *Write down the value of tan* 4θ.

Here $$\tan 4\theta = \frac{s_1 - s_3}{1 - s_2 + s_4} = \frac{4 \tan \theta - {}^4C_3 \tan^3 \theta}{1 - {}^4C_2 \tan^2 \theta + {}^4C_4 \tan^4 \theta}$$

$$= \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}.$$

Ex. *Prove that* $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

126. *By a method similar to that of the previous article it may be shewn that.*

$$\sin(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n (s_1 - s_3 + s_5 - \ldots),$$

and that

$$\cos(A_1 + A_2 + \ldots + A_n) = \cos A_1 \cos A_2 \ldots \cos A_n (1 - s_2 + s_4 - \ldots),$$

where $s_1, s_2, s_3 \ldots$ have the same values as in that article.

127. *Identities holding between the trigonometrical ratios of the angles of a triangle.*

When three angles A, B, and C, are such that their sum is 180°, many identical relations are found to hold between their trigonometrical ratios.

The method of proof is best seen from the following examples,

Ex. 1. *If* $A + B + C = 180°$,

prove that $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

$\sin 2A + \sin 2B + \sin 2C = 2\sin(A + B)\cos(A - B) + 2\sin C \cos C$

Since $A + B + C = 180°$,

we have $A + B = 180° - C$,

and therefore $\sin(A + B) = \sin C$,

and $\cos(A + B) = -\cos C$ (Art. 72)

Hence the expression

$$= 2\sin C \cos(A - B) + 2\sin C \cos C$$
$$= 2\sin C [\cos(A - B) + \cos C]$$
$$= 2\sin C [\cos(A - B) - \cos(A + B)]$$
$$= 2\sin C \cdot 2\sin A \sin B$$
$$= 4\sin A \sin B \sin C.$$

Ex. 2. *If* $A + B + C = 180°$,

prove that $\cos A + \cos B - \cos C = -1 + 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$.

The expression $= \cos A + (\cos B - \cos C)$

$$= 2\cos^2\frac{A}{2} - 1 + 2\sin\frac{B + C}{2}\sin\frac{C - B}{2}.$$

Now $B + C = 180° - A$,

so that $\dfrac{B + C}{2} = 90° - \dfrac{A}{2}$,

and therefore $\sin\dfrac{B + C}{2} = \cos\dfrac{A}{2}$

and $\cos\dfrac{B + C}{2} = \sin\dfrac{A}{2}$.

Hence the expression

$$= 2\cos^2\frac{A}{2} - 1 + 2\cos\frac{A}{2}\sin\frac{C-B}{2}$$

$$= 2\cos\frac{A}{2}\left[\cos\frac{A}{2} + \sin\frac{C-B}{2}\right] - 1$$

$$= 2\cos\frac{A}{2}\left[\sin\frac{B+C}{2} + \sin\frac{C-B}{2}\right] - 1$$

$$= 2\cos\frac{A}{2}\cdot 2\sin\frac{C}{2}\cos\frac{B}{2} - 1$$

$$= -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 3. *If* $A + B + C = 180°$,

prove that $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$.

Let $S = \sin^2 A + \sin^2 B + \sin^2 C$,

so that $2S = 2\sin^2 A + 1 - \cos 2B + 1 - \cos 2C$

$= 2\sin^2 A + 2 - 2\cos(B + C)\cos(B - C)$

$= 2 - 2\cos^2 A + 2 - 2\cos(B + C)\cos(B - C)$.

$\therefore \quad S = 2 + \cos A\,[\cos(B - C) + \cos(B + C)]$,

since $\cos A = \cos\{180° - (B + C)\} = -\cos(B + C)$.

$\therefore \quad S = 2 + \cos A\cdot 2\cos B\cos C$

$= 2 + 2\cos A\cos B\cos C$.

Ex. 4. *If* $A + B + C = 180°$,

prove that $\tan A + \tan B + \tan C = \tan A\tan B\tan C$.

By the third formula of Art. 124, we have

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - (\tan B\tan C + \tan C\tan A + \tan A\tan B)}.$$

But $\tan(A + B + C) = \tan 180° = 0$.

Hence $0 = \tan A + \tan B + \tan C - \tan A\tan B\tan C$,

i.e. $\tan A + \tan B + \tan C = \tan A\tan B\tan C$.

This may also be proved independently. For

$$\tan(A + B) = \tan(180° - C) = -\tan C.$$

$$\therefore \quad \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C.$$

$$\therefore \quad \tan A + \tan B = -\tan C + \tan A\tan B\tan C$$

i.e. $\tan A + \tan B + \tan C = \tan A\tan B\tan C$.

Ex. 5. *If* $x + y + z = xyz$,

prove that $\dfrac{2x}{1-x^2} + \dfrac{2y}{1-y^2} + \dfrac{2z}{1-z^2} = \dfrac{2x}{1-x^2}\cdot\dfrac{2y}{1-y^2}\cdot\dfrac{2z}{1-z^2}$.

Put $x = \tan A, y = \tan B$, and $z = \tan C$, so that we have

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

$$\therefore \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C,$$

so that $\tan (A + B) = \tan (\pi - C)$. [Art. 72]

Hence $A + B + C = n\pi + \pi$,

$$\therefore \quad \frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2\tan A}{1-\tan^2 A} + \frac{2\tan B}{1-\tan^2 B} + \frac{2\tan C}{1-\tan^2 C}$$

$= \tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$,

(by a proof similar to that of the previous example)

$$= \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}$$

EXAMPLES. XX.

If $A + B + C = 180°$, prove that :

1. $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C.$
2. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$
3. $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin b \cos C.$
4. $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$
5. $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$
6. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$
7. $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C.$
8. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$
9. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C.$
10. $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$
11. $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$
12. $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1.$
13. $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$
14. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$
15. $\sin (B + 2C) + \sin (C + 2A) + \sin (A + 2B)$

$$= 4 \sin \frac{B - C}{2} \sin \frac{C - A}{2} \sin \frac{A - B}{2}.$$

16. $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} - 1 = 4 \sin \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4}.$

17. $\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}=4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}.$

18. $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

19. $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$
$=4\sin A\sin B\sin C.$

If $A+B+C=2S$, prove that

20. $\sin(S-A)\sin(S-B)+\sin S\sin(S-C)=\sin A\sin B.$

21. $4\sin S\sin(S-A)\sin(S-B)\sin(S-C)$
$=1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C.$

22. $\sin(S-A)+\sin(S-B)+\sin(S-C)-\sin S$
$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

23. $\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)$
$=2+2\cos A\cos B\cos C.$

24. If $A+B+C=2S$, prove that
$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C$
$=1+4\cos S\cos(S-A)\cos(S-B)\cos(S-C).$

25. If $\alpha+\beta+\gamma+\delta=2\pi$, prove that

(1) $\cos\alpha+\cos\beta+\cos\gamma+\cos\delta+4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}=0,$

(2) $\sin\alpha-\sin\beta+\sin\gamma-\sin\delta+4\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}=0,$

and (3) $\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=\tan\alpha\tan\beta\tan\gamma\tan\delta(\cot\alpha+\cot\beta+\cot\gamma+\cot\delta).$

26. If the sum of four angles by 180°, prove that the sum of the products of their cosines taken two and two together is equal to the sum of the products of their sines taken similarly.

27. Prove that
$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=2(\sin\alpha+\sin\beta+\sin\gamma)(1+\cos\alpha+\cos\beta+\cos\gamma),$
if $\alpha+\beta+\gamma=0.$

28. Verify that
$\sin^3 a\sin(b-c)+\sin^3 b\sin(c-a)+\sin^3 c\sin(a-b)$
$+\sin(a+b+c)\sin(b-c)\sin(c-a)\sin(a-b)=0.$

If A, B, C, and D be any angles, prove that

29. $\sin A\sin B\sin(A-B)+\sin B\sin C\sin(B-C)$
$+\sin C\sin A\sin(C-A)+\sin(A-B)\sin(B-C)\sin(C-A)=0.$

30. $\sin(A-B)\cos(A+B)+\sin(B-C)\cos(B+C)$
$+\sin(C-D)\cos(C+D)+\sin(D-A)\cos(D+A)=0.$

31. $\sin(A+B-2C)\cos B-\sin(A+C-2B)\cos C$
$=\sin(B-C)\{\cos(B+C-A)+\cos(C+A-B)$
$+\cos(A+B-C)\}.$

32. $\sin(A+B+C+D)+\sin(A+B-C-D)+\sin(A+B-C+D)$
$+\sin(A+B+C-D)=4\sin(A+B)\cos C\cos D.$

33. If any theorem be true for values of A, B, and C such that
$$A+B+C=180°,$$
prove that the theorem is still true if we substitute for A, B, and C respectively the quantities

(1) $90°-\frac{A}{2}, 90°-\frac{B}{2}$, and $90°-\frac{C}{2}$,

or (2) $180°-2A, 180°-2B$, and $180°-2C$.

Hence deduce Ex. 16 from Ex. 6 and Ex. 17 from Ex. 5.

If $x+y+z=xyz$, prove that

34. $$\frac{3x-x^3}{1-3x^3}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}$$

35. $x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz$

128. *The Addition and Subtraction Theorems may be used to solve some kinds of trigonometrical equations.*

Ex. *Solve the equation*
$$\sin x+\sin 6x=\sin 3x$$
By the formulae of Art. 94, the equation is
$$2\sin 3x\cos 2x=\sin 3x.$$
$\therefore$ $\sin 3x=0$, or $2\cos 2x=1$.

If $\sin 3x=0$, then $3x=n\pi$.

If $\cos 2x=\frac{1}{2}$, then $2x=2n\pi\pm\frac{\pi}{3}$.

Hence $x=\frac{n\pi}{3}$, or $n\pi=-\frac{\pi}{6}$.

129. *To solve an equation of the form*
$$a\cos\theta+b\sin\theta=c$$
Divide both sides of the equation by $\sqrt{a^2+b^2}$, so that it may be written
$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta+\frac{b}{\sqrt{a^2+b^2}}\sin\theta=\frac{c}{\sqrt{a^2+b^2}}$$
Find from the table of tangents the angle whose tangent is $\frac{b}{a}$ and call it α.

Then $\tan\alpha=\frac{b}{a}$, so that

$$\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}, \text{ and } \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}.$$

The equation can then be written,

$$\cos\alpha\cos\theta + \sin\alpha\sin\theta = \frac{c}{\sqrt{a^2+b^2}},$$

i.e. $$\cos(\theta-\alpha) = \frac{c}{\sqrt{a^2+b^2}}.$$

Next find from the tables, or otherwise, the angle β whose cosine is $\frac{c}{\sqrt{a^2+b^2}}$,

so that $$\cos\beta = \frac{c}{\sqrt{a^2+b^2}}.$$

[N. B. This can only be done when c is $< \sqrt{a^2+b^2}$.]

The equation is then $\cos(\theta-\alpha) = \cos\beta$.

The solution of this is $\theta - \alpha = 2n\pi \pm \beta$, so that

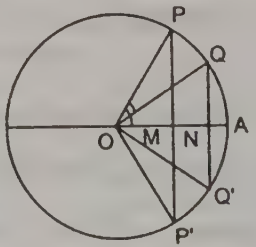
$$\theta = 2n\pi + a \pm \beta,$$

where n is any integer.

Angles, such as α and β, which are introduced into trigonometrical work to facilitate computation are called **Subsidiary Angles.**

130. The above solution may be illustrated graphically as follows :

Measure OM along the initial line equal to a, and MP perpendicular to it, and equal to b. The angle MOP is then the angle whose tangent is $\frac{b}{a}$, *i.e.* a.



With centre O and radius OP, i.e. $\sqrt{a^2+b^2}$, describe a circle, and measure ON along the initial line equal to c.

Draw QNQ' perpendicular to ON to meet the circle in Q and Q'; the angles NOQ and $Q'ON$ are therefore each equal to β.

The angle QOP is therefore $\alpha - \beta$ and $Q'OP$ is $\alpha + \beta$.

Hence the solutions of the equation are respectively

$$2n\pi + QOP \text{ and } 2n\pi + Q'OP.$$

The construction clearly fails if c be $> \sqrt{a^2+b^2}$, for then the point N would fall outside the circle.

131. As a numerical example let us solve the equation

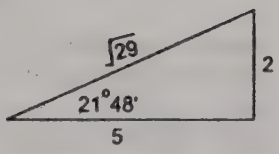
$$5\cos\theta - 2\sin\theta = 2,$$

given that $\tan 21° 48' = \frac{2}{5}$.

Dividing both sides of the equation by

$$\sqrt{5^2 + 2^2}, \text{ i.e. } \sqrt{29},$$

we have



$$\frac{5}{\sqrt{29}} \cos\theta - \frac{2}{\sqrt{29}} \sin\theta = \frac{2}{\sqrt{29}}.$$

Hence

$$\cos\theta \cos 21° 48' - \sin\theta \sin 21° 48' = \sin 21° 48' = \sin (90° - 68° 12')$$
$$= \cos 68° 12'.$$

$$\therefore \quad \cos (\theta + 21° 48') = \cos 68° 12'.$$

Hence $\theta + 21° 47' = 2n \times 180° \pm 68° 12'$. (Art. 83)

$$\therefore \quad \theta = 2n \times 180° - 21° 48' \pm 68° 12'$$
$$= 2n \times 180° - 90°, \text{ or } 2n \times 180° + 46° 24',$$

where n is any integer.

Aliter. The equation of Art. 129 may be solved in another way.

For let $t \equiv \tan\frac{\theta}{2}$.

so that $$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{2t}{1 + t^2},$$

and $$\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \frac{1 - t^2}{1 + t^2}$$ (Art. 109)

The equation then becomes

$$a\frac{1 - t^2}{1 + t^2} + b\frac{2t}{1 + t^2} = c$$

so that $$t^2 (c + a) - 2bt + c - a = 0.$$

This is a quadratic equation giving two values for t, and hence two values for $\tan\frac{\theta}{2}$.

Thus, the example of this article gives.

$$7t^2 + 4t - 3 = 0,$$

so that $$t = -1, \text{ or } \frac{3}{7}$$
$$= \tan(-45°), \text{ or } \tan 23° 12' \text{ (from the tables).}$$

Hence $\frac{\theta}{2} = n.\ 180° - 45°$, or $n\ .\ 180° + 23° 12'$,

i.e. $\theta = n.\ 360° - 90°$, or $n\ .\ 360° + 46° 24'$.

EXAMPLES XXI

Solve the equations :

1. $\sin\theta + \sin 7\theta = \sin 4\theta$.
2. $\cos\theta + \cos 7\theta = \cos 4\theta$.
3. $\cos\theta + \cos 3\theta = 2\cos 2\theta$.
4. $\sin 4\theta - \sin 2\theta = \cos 3\theta$.
5. $\cos\theta - \sin 3\theta = \cos 2\theta$.
6. $\sin 7\theta = \sin\theta + \sin 3\theta$.
7. $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.
8. $\sin\theta + \sin 3\theta + \sin 5\theta = 0$.
9. $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$.
10. $\sin(3\theta + a) + \sin(3\theta - a) + \sin(a - \theta) - \sin(a + \theta) = \cos a$.
11. $\cos(3\theta + a)\cos(3\theta - a) + \cos(5\theta + a)\cos(5\theta - a) = \cos 2a$.
12. $\cos n\theta = \cos(n - 2)\theta + \sin\theta$.
13. $\sin\frac{n+1}{2}\theta = \sin\frac{n-1}{2}\theta + \sin\theta$.
14. $\sin m\theta + \sin n\theta = 0$.
15. $\cos m\theta + \cos n\theta = 0$.
16. $\sin^2 n\theta - \sin^2(n-1)\theta = \sin^2\theta$.
17. $\sin 3\theta + \cos 2\theta = 0$
18. $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$.
19. $\sin\theta + \cos\theta = \sqrt{2}$
20. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$.
21. $\sin x + \cos x = \sqrt{2}\cos A$.
22. $5\sin\theta + 2\cos\theta = 5$ (given $\tan 21° 48' = .4$).
23. $6\cos x + 8\sin x = 9$ (given $\tan 53° 8' = 1\frac{1}{2}$ and $25° 50' = .9$)
24. $1 + \sin^2\theta = 3\sin\theta\cos\theta$ (given $\tan 71° 34' = 3$).
25. $\operatorname{cosec}\theta = \cot\theta + \sqrt{3}$.
26. $\operatorname{cosec} x = 1 + \cot x$.
27. $(2 + \sqrt{3})\cos\theta = 1 - \sin\theta$
28. $\tan\theta + \sec\theta = \sqrt{3}$
29. $\cos 2\theta = \cos^2\theta$.
30. $4\cos\theta - 3\sec\theta = \tan\theta$.
31. $\cos 2\theta + 3\cos\theta = 0$.
32. $\cos 3\theta + 2\cos\theta = 0$
33. $\cos 2\theta = (\sqrt{2} + 1)\left(\cos\theta - \frac{1}{\sqrt{2}}\right)$.
34. $\cot\theta - \tan\theta = 2$.
35. $4\cot 2\theta = \cot^2\theta - \tan^2\theta$.
36. $3\tan(\theta - 15°) = \tan(\theta + 15°)$
37. $\tan\theta + \tan 2\theta + \tan 3\theta = 0$.
38. $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$.
39. $\sin 3a = 4\sin a\sin(x + a)\sin(x - a)$.
40. Prove that the equation $x^3 - 2x + 1 = 0$ is satisfied by putting for x either of the values
$$\sqrt{2}\sin 45°, 2\sin 18°, \text{ and } 2\sin 234°.$$
41. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, prove that $\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$.
42. If $\sin(\pi\cot\theta) = \cos(\pi\tan\theta)$, prove that either $\operatorname{cosec} 2\theta$ or $\cot 2\theta$ is equal to $n + \frac{1}{4}$, where n is a positive or negative integer.

132. Ex. *To trace the changes in the expression* $\sin x + \cos x$ *as* x *increases from 0 to* 2π.

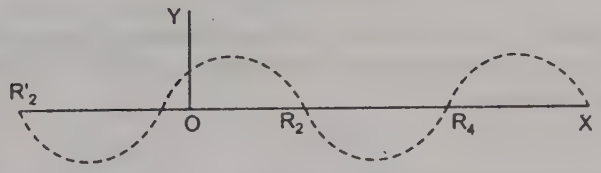
We have $\sin x = \cos x = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]$

$$= \sqrt{2}\left[\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right] = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$

We thus have the following table of values :

x	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	2π
$x + \frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{4}$	2π	$\frac{9\pi}{4}$
$\sin\left(x + \frac{\pi}{4}\right)$	$\frac{1}{\sqrt{2}}$	1	0	-1	0	$\frac{1}{\sqrt{2}}$
$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$	1	$\sqrt{2}$	0	$-\sqrt{2}$	0	1

As in Art. 62, the graph is as in the following figure.



133. Ex. *To trace the changes in the sign and magnitude of* $a\cos\theta + b\sin\theta$, *and to find the greatest value of the expression.*

We have

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right].$$

Let α be the smallest positive angles such that

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \text{ and } \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$$

The expression therefore

$$= \sqrt{a^2+b^2}\,[\cos\theta\cos\alpha + \sin\theta\sin\alpha] = \sqrt{a^2+b^2}\cos(\theta - \alpha).$$

As θ changes from α to $2\pi + \alpha$, the angle $\theta - \alpha$ changes from 0 to 2π, and hence the changes in the sign and magnitude of the expression are easily obtained.

Since the greatest value of the quantity $\cos(\theta - \alpha)$ is unity, i.e., when θ equals α, the greatest value of the expression is $\sqrt{a^2 + b^2}$. Also the value of θ which gives this greatest value is such that its cosine is $\frac{a}{\sqrt{a^2 + b^2}}$.

Similarly the expression

$$\begin{aligned} & a \sin(\theta + \alpha) + b \sin(\theta + \beta) \\ &= \sin\theta (a \cos\alpha + b \cos\beta) + \cos\theta (a \sin\alpha + b \sin\beta) \\ &= R \sin(\theta + \gamma) \qquad \ldots(1) \end{aligned}$$

where $R \cos\gamma = a \cos\alpha + b \cos\beta$,

and $R \sin\gamma = a \sin\alpha + b \sin\beta$,

so that, on squaring and adding,

$$R = \sqrt{a^2 + b^2 + 2ab \cos(\alpha - \beta)},$$

and, on division,

$$\tan\gamma = \frac{a \sin\alpha + b \sin\beta}{a \cos\alpha + b \cos\beta}.$$

The greatest value of (1), i.e., of the given expression is clearly R corresponding to the value, $\frac{\pi}{2} - \gamma$, of the angle θ.

EXAMPLES. XXII.

As θ increases from 0 to 2π, trace the changes in the sign and magnitude of the following expressions, state their periods, and plot their graphs.

1. $\sin\theta - \cos\theta$.
2. $\sin\theta + \sqrt{3}\cos\theta$.

$$\left[\text{N.B. } \sin\theta + \sqrt{3}\cos\theta = 2\left[\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right] = 2\sin\left(\theta + \frac{\pi}{3}\right).\right]$$

3. $\sin\theta - \sqrt{3}\cos\theta$.
4. $\cos^2\theta - \sin^2\theta$.
5. $\sin\theta\cos\theta$.
6. $\sin 3\theta$.
7. $\tan 3\theta$.
8. $\sec 4\theta$.
9. $\frac{\sin\theta + \sin 2\theta}{\cos\theta + \cos 2\theta}$.
10. $\sin(\pi \sin\theta)$.
11. $\cos(\pi \sin\theta)$.
12. Trace the changes in the sign and magnitude of $\frac{\sin 3\theta}{\cos 2\theta}$ as the angle increases from 0 and 90°.

CHAPTER X

LOGARITHMS

134. *SUPPOSING that we know that*

$$10^{2.4031205} = 253,\ 10^{2.6095944} = 407,$$

and $$10^{5.0127149} = 102971,$$

we can shew that $253 \times 407 = 102971$ without performing the operation of multiplication. For

$$\begin{aligned} 253 \times 407 &= 10^{2\cdot4031205} \times 10^{2\cdot6095944} \\ &= 10^{2.4031205\,+\,2.6095944} \\ &= 10^{5.0127149} = 102971. \end{aligned}$$

Here it will be noticed that the process of multiplication has been replaced by the simpler process of addition.

Again, supposing that we know that

$$10^{4.9004055} = 79507$$

and that $$10^{1.6334685} = 43,$$

we can easily shew that the cube root of 79507 is 43.

For $$\sqrt[3]{79507} = [79507]^{\frac{1}{3}} = (10^{4.9004055})^{\frac{1}{3}}$$

$$= 10^{\frac{1}{3}\,\times\,4.9004055} = 10^{1.6334685} = 43.$$

Here it will be noticed that the difficult process of extracting the cube root has been replaced by the simpler process of division.

135. Logarithm. Def. *If a be any number, and x and N two other numbers such that $a^x = N$, then x is called the logarithm of N to the base a and is written* $\log_a N$.

The logarithm of a number to a given base is therefore the index of the power to which the base must be raised that it may be equal to the given number.

Exs. Since $10^2 = 100$, therefore $2 = \log_{10} 100$.

Since $10^5 = 100000$, therefore $5 = \log_{10} 100000$.

Since $2^4 = 16$, therefore $4 = \log_2 16$.

Since $8^{\frac{2}{3}} = [8^{\frac{1}{3}}]^2 = 2^2 = 4$, therefore $\frac{2}{3} = \log_8 4$.

Since $9^{-3/2} = \frac{1}{9^{3/2}} = \frac{1}{3^3} = \frac{1}{27}$, therefore

$$-\frac{3}{2} = \log_2\left(\frac{1}{27}\right).$$

N. B. Since $a^0 = 1$ always, the logarithm of unity to any base is always zero.

136. *In Algebra, if m and n be any real quantities whatever, the following laws, known as the laws of indices, are found to be true :*

(i) $a^m \times a^n = a^{m+n}$,

(ii) $a^m \div a^n = a^{m-n}$,

and (iii) $(a^m)^n = a^{mn}$.

Corresponding to these we have three fundamental laws of logarithms, viz.

(i) $\log_a (mn) = \log_a m + \log_a n$,

(ii) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$,

and (iii) $\log_a m^n = n \log_a m$.

The proofs of these laws are given in the following articles

137. *The logarithm of the product of two quantities is equal to the sum of the logarithms of the quantities to the same base, i.e.*

$$\log_a (mn) = \log_a m + \log_a n$$

Let $x = \log_a m$, so that $a^x = m$,

and $y = \log_a n$, so that $a^y = n$.

Then $mn = a^x \times a^y = a^{x+y}$.

$\therefore$ $\log_a mn = x + y$ (Art. 135, Def.)

$= \log_a m + \log_a n$.

138. *The logarithm of the quotient of two quantities is equal to the difference of their logarithms, i.e.*

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

Let $x = \log_a m$, so that $a^x = m$, (Art. 135, Def.)

and $y = \log_a n$, so that $a^y = n$.

Then $\frac{m}{n} = a^x \div a^y = a^{x-y}$

$\therefore$ $\log_a \left(\frac{m}{n}\right) = x - y$ (Art. 135, Def.)

$= \log_a m - \log_a n$.

139. *The logarithm of a quantity raised to any power is equal to the logarithm of the quantity multiplied by the index of the power, i.e.*

$$\log_a (m^n) = n \log_a m.$$

Let $x = \log_a m$, so that $a^x = m$.

Then $m^n = (a^x)^n = a^{nx}$

$\therefore \quad \log_a(m^n) = nx$ (Art. 135, Def.)

$= n \log_a m.$

Exs. $\log 48 = \log (2^4 \times 3) = \log 2^4 + \log 3 = 4 \log 2 + \log 3;$

$$\log \frac{63}{484} = \log \frac{7 \times 3^2}{2^2 \times 11^2} = \log 7 + \log 3^2 - \log 2^2 - \log 11^2$$

$$= \log 7 + 2 \log 3 - 2 \log 2 - 2 \log 11;$$

$$\log \sqrt[5]{13} = \log 13^{1/5} = \frac{1}{5} \log 13.$$

140. Common system of logarithms : In the system of logarithms which we practically use the base is always 10, so that, if no base be expressed, the base 10 is always understood. The advantage of using 10 as the base is seen in the three following articles.

141. Characteristic and Mantissa. Def. If the logarithm of any number be partly integral and partly fractional, the integral portion of the logarithm is called its characteristic and the decimal portion is called its mantissa.

Thus, supposing that $\log 795 = 2.9003671$, the number 2 is the characteristic and .9003671 is the mantissa.

Negative characteristics. Suppose we know that

$$\log 2 = \cdot 30103.$$

Then, by Art. 138,

$$\log \frac{1}{2} = \log 1 - \log 2 = 0 - \log 2 = - \cdot 30103,$$

so that $\log \frac{1}{2}$ is negative.

Now it is found convenient, as will be seen in Art. 143, that the mantissae of all logarithms should be kept positive. We therefore instead of $- \cdot 30103$ write $- (1 - \cdot 69897)$, so that

$$\log \frac{1}{2} = - (1 - \cdot 69897) = - 1 + \cdot 69897.$$

For shortness this latter expression is written $\bar{1} \cdot 69897$.

the horizontal line over the 1 denotes that the integral part is nagative; the decimal part however is positive.

As another example, $\bar{3} \cdot 4771213$ stands for

$$- 3 + \cdot 4771213.$$

142. *The characteristic of the logarithm of any number can always be determined by inspection.*

(i) Let the number be greater than unity.

Since $10^0 = 1$, therefore $\log 1 = 0$;

since $10^1 = 10$, therefore $\log 10 = 1$;

since $10^2 = 100$, therefore $\log 100 = 2$;

and so on.

Hence the logarithm of any number lying between 1 and 10 must lie between 0 and 1, that is, it will be a decimal fraction and therefore have 0 as its characteristic.

So the logarithm of any number between 10 and 100 must lie between 1 and 2, *i.e.* it will have a characteristic equal to 1.

Similarly, the logarithm of any number between 100 and 1000 must lie between 2 and 3, *i.e.* it will have a characteristic equal to 2.

So, if the number lie between 1000 and 10000, the characteristic will be 3.

Generally, *the characteristic of the logarithm of any number will be one less than the number of digits in its integral part.*

Exs. *The number 296·3457 has 3 figures in its integral part, and therefore the characteristic of its logarithm is 2.*

The characteristic of the logarithm of 29634·57 will be $5 - 1$, *i.e.* 4.

(ii) Let the number be less than unity.

Since $10^0 = 1$, therefore $\log 1 = 0$;

since $10^{-1} = \frac{1}{10} = \cdot 1$, therefore $\log \cdot 1 = -1$;

since $10^{-2} = \frac{1}{10^2} = \cdot 01$, therefore $\log \cdot 01 = -2$;

since $10^{-3} = \frac{1}{10^3} = \cdot 001$, therefore $\log \cdot 001 = -3$;

and so on.

The logarithm of any number between 1 and ·1 therefore lies between 0 and -1, and so is equal to $-1 +$ some decimal, *i.e.* its characteristic is $\bar{1}$.

So the logarithm of any number between ·1 and ·01 lies between -1 and -2, and hence it is equal to $-2 +$ some decimal, *i.e.* its characteristics is $\bar{2}$.

Similarly, the logarithm of any number between ·01 and ·001 lies between -2 and -3, *i.e.*, its characteristic is $\bar{3}$.

Generally, *the characteristic of the logarithm of any decimal fraction will be negative and* **numerically** *will be greater by unity than the number of cyphers following the decimal point.*

For any fraction between 1 and ·1 (*e.g.* ·5) has no cypher following the decimal point, and we have seen that its characteristic is $\bar{1}$.

Any fraction between ·1 and ·01 (*e.g.* ·07) has one cypher following the decimal point, and we have seen that its characteristic is $\bar{2}$.

Any fraction between . 01 and . 001 (*e.g.* ·003) has two cyphers following the decimal point, and we have seen that its characteristic is $\bar{3}$.

Similarly for any fraction.

Exs. The characteristic of the logarithm of the number ·00835 is 3.

The characteristic of the logarithm of the number ·0000053 is $\bar{6}$.

The characteristic of the logarithm of the number ·34567 is $\bar{1}$.

143. *The mantissæ of the logarithm of all numbers, consisting of the same digits, are the same.*

This will be made clear by an example.

Suppose we are given that

$$\log 66818 = 4\cdot8248935.$$

Then

$$\log 668\cdot18 = \log\frac{66818}{100}$$
$$= \log 66818 - \log 100 \quad \text{(Art. 138)}$$
$$= 4\cdot8248935 - 2 = 2\cdot8248935;$$

$$\log\cdot66818 = \log\frac{66818}{100000} = \log 66818 - \log 100000 \quad \text{(Art. 138)}$$
$$= 4\cdot8248935 - 5 = \bar{1}\cdot8248935.$$

So
$$\log\cdot00066818 = \log\frac{66818}{10^8} = \log 66818 - \log 10^8$$
$$= 3\cdot8248935 - 8 = \bar{4}\cdot8248935.$$

Now the numbers 66818, 668·18, ·66818, and ·00066818 consist of the same significant figures, and only differ in the position of the decimal point. We observe that their logarithms have the same decimal portion, i.e. the same mantissa, and they only differ in the characteristic.

The value of this characteristic is in each case determined by the rule of the previous article.

It will be noted that the mantissa of a logarithm is always positive.

144. Tables of logarithms : The logarithms of all numbers from 1 to 108000 are given in chambers' Tables of Logarithms. Their values are there given correct to seven places of decimals.

The student should have access to a copy of the above table of logarithms or to some other suitable table. It will be required for many examples in the course of the next few chapters.

On the opposite page in a specimen, page selected from chambers' Tables. It gives the mantissae of the logarithms of all whole numbers from 52500 to 53000.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
5250	720 1593	1676	1758	1841	1924	2007	2089	2172	2255	2337	
51	2420	2503	2586	2668	2751	2834	2916	2999	3082	3164	
52	3247	3330	3413	3495	3578	3661	3743	3826	3909	3991	
53	4074	4157	4239	4322	4405	4487	4570	4653	4735	4818	
54	4901	4983	5066	5149	5231	5314	5397	5479	5562	5645	
55	5727	5810	5892	5975	6058	6140	6223	6306	6388	6471	
56	6554	6636	6719	6801	6884	6967	7049	7132	7215	7297	
57	7380	7462	7545	7628	7710	7793	7875	7958	8041	8123	
58	8206	8288	8371	8454	8536	8619	8701	8784	8867	8949	
59	9032	9114	9197	9279	9362	9445	9527	9610	9692	9775	
60	9857	9940	0023	0105	0188	0270	0353	0435	0518	0600	
5261	721 0683	0766	0848	0931	1013	1096	1178	1261	1343	1426	
62	1508	1591	1674	1756	1839	1921	2004	2086	2169	2251	
63	2334	2416	2499	2581	2664	2746	2829	2911	2994	3076	
64	3159	3241	3324	3406	3489	3571	3654	3736	3819	3901	
65	3984	4066	4149	4231	4314	4396	4479	4561	4644	4726	
66	4809	4891	4973	5056	5138	5221	5303	5386	5468	5551	
67	5633	5716	5798	5881	5963	6045	6128	6210	6293	6375	
68	6458	6540	6623	6705	6787	6870	6952	7035	7117	7200	
69	7282	7364	7447	7529	7612	7694	7777	7859	7941	8024	
70	8106	8189	8271	8353	8436	8518	8601	8683	8765	8848	
5271	8930	9013	9095	9177	9260	9342	9424	9507	9589	9672	82
72	9754	9836	9919	0001	0084	0166	0248	0331	0413	0495	1 8
73	722 0578	0660	0742	0825	0907	0990	1072	1154	1237	1319	2 16
74	1401	1484	1566	1648	1731	1813	1895	1978	2060	2142	3 25
75	2225	2307	2389	2472	2554	2636	2719	2801	2883	2966	4 33
76	3048	3130	3212	3295	3377	3459	3542	3624	3706	3789	5 41
77	3871	3953	4036	4118	4200	4282	4365	4447	4529	4612	6 49
78	4694	4776	4858	4941	5023	5105	5188	5270	5352	5434	7 57
79	5517	5599	5681	5763	5846	5928	6010	6092	6175	6257	8 66
80	6339	6421	6504	6586	6668	6750	6833	6915	6997	7079	9 74
5281	7162	7244	7326	7408	7491	7573	7655	7737	7820	7902	
82	7984	8066	8148	8231	8313	8395	8477	8559	8642	8724	
83	8806	8888	8971	9053	9135	9217	9299	9382	9464	9546	
84	9628	9710	9792	9875	9957	0039	0121	0203	0286	0368	
85	723 0450	0532	0614	0696	0779	0861	0943	1025	1107	1189	
86	1272	1354	1436	1518	1600	1682	1765	1847	1929	2011	
87	2093	2175	2257	2340	2422	2504	2586	2668	2750	2832	
88	2914	2996	3079	3161	3243	3325	3407	3489	3571	3654	
89	3736	3818	3900	3982	4064	4146	4228	4310	4393	4475	
90	4557	4639	4721	4803	4885	4967	5049	5131	5213	5296	
5291	5378	5460	5542	5624	5706	5788	5870	5952	6034	6116	
92	6198	6280	6362	6445	6527	6609	6691	6773	6855	6937	
93	7019	7101	7183	7265	7347	7429	7511	7593	7675	7757	
94	7839	7921	8003	8085	8167	8250	8332	8414	8496	8578	
95	8660	8742	8824	8906	8988	9070	9152	9234	9316	9398	
96	9480	9562	9644	9726	9808	9890	9972	0054	0136	0218	
97	724 0300	0382	0464	0546	0628	0710	0792	0874	0956	1038	
98	1120	1202	1283	1365	1447	1529	1611	1693	1775	1857	
99	1939	2021	2103	2185	2267	2349	2431	2513	2595	2677	
5300	2759	2841	2923	3005	3086	3168	3250	3332	3414	3496	

A table of logarithms, to five places of decimals, is given at the end of this book. It will be found to be sufficiently accurate for most practical purposes.

145. To obtain the logarithm of any such number, such as 52687, we proceed as follows. Run the eye down the extreme left-hand column until it arrives at the number 5268. then look horizontally until the eye sees the figures 7035 which are vertically beneath the number 7 at the top of the page. The number corresponding to 52687 is therefore 7217035. But this last number consists only of the digits of the mantissa, so that the mantissa required is ·7217035. Also the characteristic for 52687 is 4.

Hence $\log 52687 = 4 \cdot 7217035.$

So $\log \cdot 52687 = \bar{1} \cdot 7217035,$

and $\log \cdot 00052687 = \bar{4} \cdot 7217035.$

If, again, the logarithm of 52725 be required, the student will find (on running his eye vertically down the extreme left-hand column as far as 5272 and then horizontally along the row until it comes to the column under the digit 5) the number $\overline{0166}$. The bar which is placed over these digits denotes that to them must be prefixed not 721 but 722. Hence the mantissa corresponding to the number 52725 is ·7220166.

Also the characteristic of the logarithm of the number 52725 is 4.

Hence $\log 52725 = 4 \cdot 7220166.$

So $\log \cdot 052725 = \bar{2} \cdot 7220166$

If we use the five-figure tables at the end of this book we proceed thus : Run the eye down the extreme left-hand column of pages xxii and xxiii until it arrives at the number 52; then look horizontally until the eye sees the figures 72099 below the figure 6 at the top of the previous page. We thus get

No.	log
526	2·72099

The next figure in our number is 8. Look at the extreme right of the page of logarithms for the Mean Differences. Run the eye down the column headed by 8 until it arrives at the horizontal line commencing with 52, and there we find the number 66.

Our last figure is 7 and the corresponding number in the Difference column is 58. We thus have

	No.	log
	526	2·72099
diff. for	· 8	66
diff. for	·07	58
	526·87	2·721708

Hence, as before, $\log 52687 = 4 \cdot 72171$, correct to five places.

This is as accurate a result as we can get from using a five-figure table with the corresponding table of Mean Differences. It will be noticed that it differs in the fifth place of decimals from the value found above.

We shall now work a few numerical examples to shew the efficiency of the application of logarithms for purposes of calculation.

146. Ex. 1. *Find the value of* $\sqrt[5]{23\cdot4}$

Let $x = \sqrt[5]{23\cdot4} = (23\cdot4)^{\frac{1}{5}}$,

so that $\log x = \frac{1}{5}\log(23\cdot4)$ by Art. 139.

In a seven-figure table of logarithms we find, opposite the number 234, the logarithm 3692159.

Hence $\log 23\cdot4 = 1\cdot3692159$

Therefore $\log x = \frac{1}{5}[1\cdot3692159] = \cdot2738432.$

Again, in the table of logarithms we find, corresponding to the logarithm 2738432, the number 187864, so that

$$\log 1\cdot87864 = \cdot2738432.$$

$$\therefore \quad x = 1\cdot87864.$$

Ex. 2. *Using the tables at the end of this book. Find the value of*

$$\frac{(6\cdot45)^3 \times \sqrt[3]{\cdot00034}}{(9\cdot37)^2 \times \sqrt[4]{8\cdot93}}$$

Let x be the required value so that, by Arts. 138 and 139.

$$\log x = \log(6\cdot45)^3 + \log(\cdot00034)^{\frac{1}{3}} - \log(9\cdot37)^2 - \log(8\cdot93)^{\frac{1}{4}}$$

$$= 3\log(6\cdot45) + \frac{1}{3}\log(\cdot00034) - 2\log(9\cdot37) - \frac{1}{4}\log 8\cdot93.$$

Now in the table of logarithms of pages xxiii to xxv we find
corresponding to the number 645 the logarithm 80956
corresponding to the number 34 the logarithm 53148
corresponding to the number 937 the logarithm 97174
and corresponding to the number 893 the logarithm 95085
Hence

$$\log x = 3 \times \cdot80956 + \frac{1}{3}(4\cdot53148) - 2 \times \cdot97174 - \frac{1}{4} \times \cdot95085$$

But $\frac{1}{3}(\bar{4}\cdot53148) = \frac{1}{3}[\bar{6} + 2\cdot53148]$

$$= \bar{2} + \cdot843827.$$

$$\therefore \quad \log x = 2\cdot42868 + [\bar{2} + \cdot843827] - 1\cdot94348 - \cdot237713$$

$$= 3\cdot272507 - 4\cdot181193$$

$$= \bar{1} + 4\cdot272507 - 4\cdot181193$$

$$= \bar{1}\cdot09131, \text{ correct to five places of decimals.}$$

In the table of logarithms on page xxii we find, corresponding to the number 123, the logarithm ·08991, so that we have

$$\log 123 = 2\cdot08991$$

$$\text{diff. for } \cdot4 = 140$$

$$\log 123\cdot4 = 2\cdot09131$$

Hence $\log x = \log\cdot12340,$

and therefore $x = \cdot12340.$

When the logarithm of any number does not quite agree with any logarithm in the tables, but lies between two consecutive logarithms, it will be shewn in the next chapter how the number may be accurately found.

Ex. 3. *Having given* $\log 2 = \cdot 30103$, *find the number of digits in* 2^{67} *and the position of the first significant figure in* 2^{-37}.

We have $\log 2^{67} = 67 \times \log 2 = 67 \times \cdot 30103$
$= 20 \cdot 16901.$

Since the characteristic of the logarithm of 2^{67} is 20, it follows, by Art. 142, that in 2^{67} there are 21 digits.

Again, $\log 2^{-37} = -37 \log 2 = -37 \times \cdot 30103$
$= -11 \cdot 13811 = \overline{12} \cdot 86189.$

Hence, by Art. 142, in 2^{-37} there are 11 cyphers following the decimal-point, *i.e.* the first significant figure is in the twelfth place of decimals.

Ex. 4. *Given* $\log 3 = \cdot 4771213$, $\log 7 = \cdot 8450980$, *and* $\log 11 = 1 \cdot 0413927$, *solve the equation*

$$3^x \times 7^{2x+1} = 11^{x+5}.$$

Taking logarithms of both sides, we have

$$\log 3^x + \log 7^{2x+1} = \log 11^{x+5}.$$

$\therefore \quad x \log 3 + (2x + 1) \log 7 = (x + 5) \log 11.$

$\therefore \quad x[\log 3 + 2 \log 7 - \log 11] = 5 \log 11 - \log 7.$

$\therefore$

$$x = \frac{5 \log 11 - \log 7}{\log 3 + 2 \log 7 - \log 11}$$

$$= \frac{5 \cdot 2069635 - \cdot 8450980}{\cdot 4771213 + 1 \cdot 6901960 - 1 \cdot 0413927}$$

$$= \frac{4 \cdot 3618655}{1 \cdot 1259246} = 3 \cdot 87 \ldots .$$

147. *To prove that*

$$\log_a m = \log_b m \times \log_a b.$$

Let $\log_a m = x$, so that $a^x = m$

Also let $\log_b m = y$, so that $b^y = m$.

$\therefore \quad a^x = b^y.$

Hence $\log_a (a^x) = \log_a (b^y).$

$\therefore \quad x = y \log_a b.$ (Art. 139.)

Hence $\log_a m = \log_b m \times \log_a b.$

By the theorem of the foregoing article we can from the logarithm of any number to a base b find its logarithm to any other base a. It is found convenient, as will appear in Part II, Art. 12, not to calculate the logarithms to base 10 directly, but to calculate them first to another base and then to transform them by this theorem.

EXAMPLES. XXIII.

1. Given $\log 4 = \cdot 60206$ and $\log 3 = \cdot 4771213$, find the logarithms of $\cdot 8$, $\cdot 003$, $\cdot 0108$, and $(\cdot 00018)^{1/7}$.
2. Given $\log 11 = 1 \cdot 0413927$ and $\log 13 = 1 \cdot 1139434$, find the values of (1) $\log 1 \cdot 43$, (2) $\log 133 \cdot 1$, (3) $\log \sqrt[4]{143}$, and (4) $\log \sqrt[3]{\cdot 00169}$.
3. What are the characteristics of the logarithms of $243 \cdot 7$, $\cdot 0153$, $2 \cdot 8713$, $\cdot 00057$, $\cdot 023$, $\sqrt[5]{24615}$ and $(24589)^{3/4}$?
4. Find the 5th root of $\cdot 003$, having given
   $\log 3 = \cdot 4771213$ and $\log 312936 = 5 \cdot 4954243$.
5. Find the values of (1) $7^{1/7}$, (2) $(84)^{2/5}$, and (3) $(\cdot 021)^{1/5}$, having given
   $\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$,
   $\log 7 = \cdot 8450980$, $\log 132057 = 5 \cdot 1207283$,
   $\log 588453 = 5 \cdot 7697117$,
   and $\log 461791 = 5 \cdot 6644438$
6. Having given $\log 3 = \cdot 4771213$,
   find the number of digits in
   (1) 3^{43}, (2) 3^{27}, and (3) 3^{62}
   and the position of the first significant figure in
   (4) 3^{-13}, (5) 3^{-43}, and (6) 3^{-65}
7. Given $\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$, and $\log 7 = \cdot 8450980$; solve the equations
   (1) $2^x \cdot 3^{x+4} = 7^x$, (2) $2^{2x+1} \cdot 3^{3x+2} = 7^{4x}$,
   (3) $7^{2x} \div 2^{x-4} = 3^{3x-7}$ and (4) $\left.\begin{matrix} 7^{x+y} \times 3^{2x+y} = 9 \\ 3^{x-y} \div 2^{x-2y} = 3^x \end{matrix}\right\}$.
8. From the tables find the seventh root of $\cdot 000026751$.
   Making use of the tables, find the approximate values of :
9. $\sqrt[3]{645 \cdot 3}$.
10. $\sqrt[5]{82357}$.
11. $\dfrac{\sqrt{5} + \sqrt[3]{7}}{\sqrt[4]{8} \times \sqrt[5]{9}}$.
12. $\sqrt[3]{\dfrac{7 \cdot 2 \times 8 \cdot 3}{9 \cdot 4 \div 16 \cdot 5}}$.
13. $\sqrt{\dfrac{8\frac{1}{5} \times 11\frac{1}{3}}{\sqrt{74} \times \sqrt[5]{62}}}$

   Draw the graphs of
14. $\log x$.
15. $\log \sin x$.
16. $\log \cos x$.
17. $\log \tan x$.
18. $\log \operatorname{cosec} x$.
19. $\log \cot x$.

CHAPTER XI

THE PRINCIPLE OF PROPORTIONAL PARTS

148. We have pointed out that the logarithms of all numbers from 1 to 108000 may be found in Chambers, Mathematical Tables, so that, for example, the logarithms of 74583 and 74584 may be obtained directly therefrom.

Suppose however we wanted the logarithm of a number lying between these two, *e.g.* the number 74583·3.

To obtain the logarithm of this number we use the Principle of Proportional Parts which states that the increase in the logarithm of a number is proportional to the increase in the number itself.

Thus from the tables we find

$$\log 74583 = 4\cdot8726398 \qquad ...(1)$$

and $$\log 74584 = 4\cdot8726457 \qquad ...(2)$$

The quantity $\log 74583\cdot3$ will clearly lie between $\log 74583$ and $\log 74584$.

Let then $$\log 74583\cdot3 = \log 74583 + x = 4\cdot8726398 + x \qquad ...(3)$$

From (1) and (2), we see that for an increase 1 in the number the increase in the logarithm is ·0000059.

The Theory of Proportional Parts then states that for an increase of ·3 in the number the increase in the logarithm is

$$\cdot3 \times \cdot0000059, \text{ i.e., } \cdot00000177.$$

Hence $$\log 74583\cdot3 = 4\cdot8726398 + \cdot00000177$$

$$= 4\cdot87264157.$$

149. *As another example, we shall find the value of* log · 0382757 *and shall exhibit the working in a more concise form.*

From the tables we obtain

$$\log \cdot038275 = \bar{2}\cdot5829152$$

$$\log\cdot038276 = \bar{2}\cdot5829265.$$

Hence the difference for

$$\cdot000001 = \cdot0000113.$$

Therefore the difference for

$$\cdot0000007 = \cdot7 \times \cdot0000113$$

$$= 00000791$$

$$\therefore \quad \log \cdot 0382757 = \bar{2} \cdot 5829152$$
$$+ \cdot 00000791$$
$$\bar{2} \cdot 58292311.$$

Since the logarithms we are using are only correct to seven places of decimals, we omit the last digit and the answer is $\bar{2} \cdot 5829231$.

150. The converse question is often met with, viz., to find the number whose logarithm is given. If the logarithm be one of those tabulated, the required number is easily found. The method to be followed when this is not the case is shewn in the following example.

Ex. *Find the number whose logarithm is* $2 \cdot 6283924$.

On reference to the tables we find that the logarithm 6283924 is not tabulated, but that the nearest logarithms are 6283889 and 6283991 between which our logarithm lies.

We have then $\log 425 \cdot 00 = 2 \cdot 6283889$ ...(1)

and $\log 425 \cdot 01 = 2 \cdot 6283991$ ...(2)

Let $\log (425 \cdot 00 + x) = 2 \cdot 6283924$ ...(3)

From (1) and (2), we see that corresponding to a difference of $\cdot 01$ in the number there is a different of $\cdot 0000102$ in the logarithm.

From (1) and (3), we see that corresponding to a difference of x in the number there is a difference of $\cdot 0000035$ in the logarithm.

Hence we have $x : \cdot 01 :: \cdot 0000035 : \cdot 0000102$.

$$\therefore \quad x = \frac{35}{102} \times \cdot 01 = \frac{\cdot 35}{102} = \cdot 00343 \text{ nearly.}$$

Hence the required number $= 425 \cdot 00 + \cdot 00343 = 425 \cdot 00343$.

151. Where logarithms are taken out of the tables the labour of subtracting successive logarithms may be avoided. On reference to page 153 there is found at the extreme right a column headed *Diff.* The number 82 at the head of the figures in this column gives the difference corresponding to a difference of unity in the numbers on that page.

This number 82 means $\cdot 0000082$.

The rows below the 82 give the differences corresponding to $\cdot 1, \cdot 2, \ldots$ Thus the fifth of these rows means that the difference for $\cdot 5$ is $\cdot 0000041$.

As an example, let us find the logarithm of $52746 \cdot 74$

From page 153, we have

$$\log 52746 = 4 \cdot 7221895$$
$$\text{diff. for } \cdot 7 = \cdot 0000057$$
$$\text{diff. for } \cdot 04 \left(= \frac{1}{10} \times \text{diff. for } 4 \right) = \cdot 0000003$$

$$\therefore \quad \log 52746 \cdot 74 = 4 \cdot 7221955.$$

We shall solve two more examples, taking all the logarithms from the tables, and only putting down the necessary steps.

Ex. 1. *Find the seventh root of* ·034574.

If x be the required quantity, we have

$$\log x = \frac{1}{7}\log(\cdot 034574) = \frac{1}{7}(\bar{2}\cdot 5387496)$$

$$= \frac{1}{7}(7 + 5\cdot 5387496).$$

$$\therefore \quad \log x = \bar{1}\cdot 7912499.$$

But $\log \cdot 61837 = \bar{1}\cdot 7912484$

diff. = ·0000015.

But diff. for ·00001 = ·0000071,

$\therefore$ required increase = ·00000211,

$\therefore \quad x = \cdot 61837211$

71) 150 (2.11
142
80
71
90
71
19

Ex. 2. *If* $a = 345\cdot 75$ *and* $b = 283\cdot 912$, *find the value of square root of* $a^2 - b^2$, *using the five-figure tables at the end of this book.*

If x be the required quantity, we have

$$2\log x = \log(a^2 - b^2) = \log(a - b) + \log(a + b)$$
$$= \log 61\cdot 838 + \log 629\cdot 662.$$

Now, from page xxiv,

log 61·8		=	1·79099	
diff. for	·03	=	21	
diff. for	·008	=	5	6
log 629		=	2·79865	
diff. for	·6	=	42	
diff. for	·06	=	4	2
diff. for	·002	=		1 4

Hence, by addition, $2\log x = 4\cdot 59037$, correct to 5 places.

$\therefore \log x = 2\cdot 29518$ | 5

But, from page xxii,

log 197		=	2·29447
diff. for	·3	=	67
diff. for	·02	=	4	5

$\therefore \log 197\cdot 32 = 2\cdot 29518$ | 5

$\therefore \quad x = 197\cdot 32$

If we ussed seven-figure logarithms we should obtain a more accurate result, viz. 197·3247.

152. The proof of the Principle of Proportional Parts will not be given at this stage. It is not strictly true without certain limitations.

If we use seven-figure tables, the numbers to which the principle is applied must contain not less than five significant figures, and then we may rely on the result as correct to seven places of decimals.

For example, we must *not* aply the principle to obtain the value of $\log 2{\cdot}5$ from the values of $\log 2$ and $\log 3$.

For, if we did, since these logarithms are ·30103 and ·4771213, the logarithm of $2{\cdot}5$ would be ·389075.

But from the tables the value of $\log 2{\cdot}5$ is found to be ·3979400.

Hence the result which we should obtain would be manifestly quite incorrect.

So, if the numbers to which the principle is applied contain not less than four digits, we can rely on the result as correct to five places of decimals.

Similarly, if the numbers contain not less than three digits, the result can be relied upon in general to four places of decimals.

Tables of trigonometrical ratios.

153. In chambers' Tables will be found tables giving the values of the trigonometrical ratios of angles between 0° and 45°, the angles increasing by differences of 1′.

It is unnecessary to separately tabulate the ratios for angles between 45° and 90°, since the ratios of angles betwen 45° and 90° can be reduced to those of angles between 0° and 45°. (Art. 75)

[For example,

$$\sin 76° 11' = \sin (90° - 13° 49') = \cos 13° 49',$$

and is therefore known.]

Such a table is called a table of natural sines, cosines, etc. to distinguish it from the table of logarithmic sines, cosines, etc.

If we want to find the sine of an angle which contains an integral number of degrees and minutes, we can obtain it from the tables. If, however, the angle contian seconds, we must use the principle of proportional parts.

Ex. 1. *Given* $\sin 29° 14' = 4883674$,

and $\sin 29° 15' = \cdot 4886212$

find the value of $\sin 29° 14' 32''$.

By subtraction we have

$$\text{difference in the sine for } 1' = \cdot 0002538.$$

$$\therefore \quad \text{difference in the sine for } 32'' = \frac{32}{60} \times \cdot 0002538 = \cdot 00013536,$$

$$\therefore \quad \sin 29° 14' 32'' = \cdot 4883674$$

$$\underline{\qquad + \cdot 00013536}$$

$$\cdot 48850276.$$

Since we want our answer only to seven places of decimals, we omit the last 6, and, since 76 is nearer to 80 than 70, we write

$$\sin 29° 14' 32'' = \cdot 4885028.$$

When we omit a figure in the eighth place of decimals we add 1 to the figure in the seventh place, if the omitted figure be 5 or a number greater than 5.

Ex. 2. *Given* $\cos 16° 27' = \cdot 9590672$,
and $\cos 16° 28' = \cdot 9589848$,
find the value of $\cos 16° 27' 47''$.

We note that, as was shewn in Art. 55, the cosine decreases as the angle increases.

Hence for an increase of $1'$ *i.e.* $60''$. in the angle, there is a decrease of $\cdot 0000824$ in the cosine.

Hence for an increase of $47''$ in the angle, there is a decrease of $\frac{47}{60} \times \cdot 0000824$ in the cosine.

$$\therefore \quad \cos 16° 27' 47'' = \cdot 9590672 - \frac{47}{60} \times \cdot 0000824$$
$$= \cdot 9590672 - \cdot 0000645$$
$$= \cdot 9590672$$
$$= \cdot 0000645$$
$$= \cdot 9590027$$

In practice this may be abbreviated thus :

$\cos 16° 28' = \cdot 9589848$
$\cos 16° 27' = \cdot 9590672$
diff. for $1' = - \cdot 0000824.$
$\therefore$ diff. for $47'' = \frac{4\,7}{6\,0} \times \cdot 0000824$
$= - \cdot 0000645.$
$\therefore$ Ans. $= \cdot 9590672$
$= \frac{- \cdot 0000645}{\cdot 9590027}.$

824
47
5768
3296
60) 38728
645

154. An example of the inverse question, to find the angle when one of its trigonometrical ratios is given, will now be given.

Ex. *Find the angle whose cotangent is* $1 \cdot 4109325$, *having given* $\cot 35° 19' = 1 \cdot 4114799$, and $\cot 35° 20' = 1 \cdot 4106098$.

Let the required angle be $35° 19' + x''$,
so that $\cot (35° 19' + x'') = 1 \cdot 4109325$.
From these three equations we have
For an increase of $60''$ in the angle, a decrease of $\cdot 0008701$ in the cotangent,
For an increase of x'' in the angle, a decrease of $\cdot 0005474$ in the cotangent,

$$\therefore x : 60 :: 5474 : 8701, \text{ so that } x = 37 \cdot 7$$

Hence the required angle $= 35° 19' 37 \cdot 7''$.

155. In working all questions involving the application of the Principle of Proportional Parts, the student must be very careful to note whether the trigonometrical ratios increases or decrease as the angle increases. As a help to his memory, he may observe that in the first quadrant the three trigonometrical ratios whose names begin with co-, *i.e.* the cosine, the cotangent, and the cosecant, all decrease as the angle increases.

Tables of logarithmic sines, cosines, etc.

156. In many kinds of trigonometric calculation, as in the solution of triangles, we often require the logarithms of trigonometrical ratios. To avoid the inconvenience of first finding the sine of any angle from the tables and then obtaining the logarithm of this sine by a second application of the tables, it has been found desirable to have separate tables giving the logarithms of the various trigonometrical functions of angles. As before, it is only necessary to construct the tables for angles between 0° and 45°

Since the sine of an angle is always less than unity, the logarithm of its sine is always negative (Art. 142).

Again, since the tangent of an angle between 0° and 45° is less than unity its logarithm is negative, whilst the logarithm of the tangent of an angle between 45° and 90° is the logarithm of a number greater than unity and is therefore positive.

157. To avoid the trouble and inconvenience of printing the proper sign to the logarithms of the trigonometric functions, the logarithms as tabulated are not the true logarithms, but the true logarithms *increased by* 10.

For example, $\sin 30° = \frac{1}{2}$.

Hence $\log \sin 30° = \log \frac{1}{2} = -\log 2$

$= - \cdot 30103 = \bar{1} \cdot 69897.$

The logarithm tabulated is therefore

$10 + \log \sin 30°$ *i.e.* $9 \cdot 69897.$

Again, $\tan 60° = \sqrt{3}.$

Hence $\log \tan 60° = \frac{1}{2} \log 3 = \frac{1}{2} (\cdot 4771213)$

$= \cdot 2385606.$

The logarithm tabulated is therefore

$10 + \cdot 2385606$, *i.e.* $10 \cdot 2385606.$

The symbol L is used to denote these "tabular logarithms," *i.e.* the logarithms as found in the English books of tables.

Thus $L \sin 15° 25' = 10 + \log \sin 15° 25'$,

and $L \sec 48° 23' = 10 + \log \sec 48° 23'$.

158. If we want to find the tabular logarithm of any function of an angle, which contains an integral number of degrees and minutes, we can obtain it directly from the tables. If, however, the angle contains seconds we must use the principle of proportional parts. The method of procedure is similar to that of Art. 153. We give an example and also one of the inverse question.

Ex. 1. *Given* $L \operatorname{cosec} 32° 21' = 10 \cdot 2715733,$

and $L \operatorname{cosec} 32° 22' = 10 \cdot 2713740,$

find $L \operatorname{cosec} 32° 21' 51''$

For an increase of 60'' in the angle, there is a *decrease* of ·0001993 in the logarithm.

Hence for an increase of 51'' in the angle, the corresponding *decrease* is $\frac{51}{60} \times$ ·0001993, *i.e.* ·0001694.

Hence L cosec 32° 21′ 51″ = 10·2715733
$$= \frac{- \cdot 0001694}{10 \cdot 2714039}.$$

Ex. 2. *Find the angle such that the tabular logarithm of its tangent is* 9·4417250.

Let x be the required angle.

From the tables, we have

$L \tan x = 9 \cdot 4417250$
$L \tan 15° 27' = 9 \cdot 4415145$

diff. = 2105

$L \tan 15° 28' = 9 \cdot 4420062$
$L \tan 15° 27' = 9 \cdot 4415145$

diff. for 1′ = 4917.

Corresponding increase
$$= \frac{2105}{4917} \times 60''$$
$$= 25 \cdot 7''$$
$\therefore \quad x = 15° 27' 26''$

	2105
	60
4917)	126300 (25.7
	9834
	27960
	24585
	33750

Ex. 3. *Given* $L \sin 14° 6' = 9 \cdot 3867040$, find L cosec 14° 6′,

Here $\log \sin 14° 6' = L \sin 14° 6' - 10$
$= - 1 + \cdot 3867040,$

Now $\log \operatorname{cosec} 14° 6' = \log \frac{1}{\sin 14° 6'}$
$= - \log \sin 14° 6'$
$= 10 - L \sin 14° 6'$
$= 10 - 9 \cdot 3867040$
$= \cdot 6132960.$

Hence L cosec 14° 6′ = 10·6132960.

More generally, we have
$$\sin \theta \times \operatorname{cosec} \theta = 1.$$
$\therefore \quad \log \sin \theta + \log \operatorname{cosec} \theta = 0.$
$\therefore \quad L \sin \theta + L \operatorname{cosec} \theta = 20.$

The error to be avoided is this; the student sometimes assumes that, because
$$\log \operatorname{cosec} 14° 6' = - \log \sin 14° 6'.$$
he may therefore assume that
$$L \operatorname{cosec} 14° 6' = - L \sin 14° 6'$$
This is obviously untrue.

EXAMPLES. XXIV

1. Given $\log 35705 = 4{\cdot}5527290$
and $\log 35706 = 4{\cdot}5527412$,
find the values of $\log 35705{\cdot}7$ and $\log 35{\cdot}70585$.

2. Given $\log 5{\cdot}8742 = {\cdot}7689487$
and, $\log 587{\cdot}43 = 2{\cdot}7689561$
find the value of $\log 58742{\cdot}57$ and $\log {\cdot}00587422$.

3. Given $\log 47847 = 4{\cdot}6798547$
and $\log 47848 = 4{\cdot}6798638$,
find the numbers whose logarithms are respectively
$2{\cdot}6798593$ and $\bar{3}{\cdot}6798617$.

4. Given $\log 258{\cdot}36 = 2{\cdot}4122253$
and, $\log 2{\cdot}5837 = {\cdot}4122421$,
find the numbers whose logarithms are
${\cdot}4122378$ and $\bar{2}{\cdot}4122287$.

5. From the table on page 153 find the logarithms of
(1) 52538·97, (2) 527·286, (3) ·000529673,
and the numbers whose logarithms are
(4) 3·7221098, (5) $\bar{2}{\cdot}7240075$, and (6) ·7210386.

6. Given $\sin 43° 23' = {\cdot}6868761$
and $\sin 43° 24' = {\cdot}6870875$,
find the value of $\sin 43° 23' 47''$.

7. Find also the angle whose sine is ·6870349.

8. Given $\cos 32° 16' = {\cdot}8455726$
and $\cos 32° 17' = {\cdot}8454172$,
find the values of $\cos 32° 16' 24''$ and of $\cos 32° 16' 47''$.

9. Find also the angles whose cosines are
·8454832 and ·8455176.

10. Given $\tan 76° 21' = 4{\cdot}1177784$
and $\tan 76° 22' = 4{\cdot}1230079$,
find the values of $\tan 76° 21' 29''$ and $\tan 76° 21' 47''$.

11. Given $\operatorname{cosec} 13° 8' = 4{\cdot}4010616$
and $\operatorname{cosec} 13° 9' = 4{\cdot}3955817$.
find the values of $\operatorname{cosec} 13° 8' 19''$ and $\operatorname{cosec} 13° 8' 37''$.

12. Find also the angle whose cosecant is 4·396789.

13. Given $L \cos 34° 44' = 9{\cdot}9147729$
and $L \cos 34° 45' = 9{\cdot}9146852$,
find the value of $L \cos 34° 44' 27''$.

14. Find also the angle θ, where
$L \cos \theta = 9{\cdot}9147328$.

15. Given $L \cot 71° 27' = 9·5257779$
and $L \cot 71° 28' = 9·5253589$,
find the value of $L \cot 71° 27' 47''$
and solve the equation $L \cot \theta = 9·5254782$.

16. Given $L \sec 18° 27' = 10·0229168$
and $L \sec 18° 28' = 10·0229590$.
find the value of $L \sec 18° 27' 35''$.

17. find also the angle whose $L \sec$ is $10·0229285$.

18. Find in degrees, minutes and seconds the angle whose sine is $·6$, given that

$$\log 6 = ·7781513,\ L \sin 36° 52' = 9·7781186,$$

and $L \sin 36° 53' = 9·7782870$.

159. On the next page is printed a specimen page taken from Chambers' Tables. It gives the tabular logarithms of the trigonometrical ratios of angles between 32° and 33° and also between 57° and 58°.

The first column gives the L sine for each minute between 32° and 33°.

In the second column under the word Diff. is found the number 2021. This means that $·0002021$ is the difference between $L \sin 32° 0'$ and $L \sin 32° 1'$; this may be verified by subtracting $9·7242097$ from $9·7244118$. It will also be noted that the figures 2021 are printed half-way between the numbers $9·7242097$ and $9·7244118$, thus clearly shewing between what numbers it is the difference.

This same column of Differences also applies to the column on its right-hand side which is headed Cosec.

Similarly the fifth column, which is also headed Diff., may be used with the two columns on the right and left of it,

160. There is one point to be noticed in using the columns headed Diff. It has been pointed out that 2021 (at the top of the second column) means $·0002021$. Now the 790 (at the top of the eighth column) means *not* $·000790$, but $·0000790$. The rule is this; the right-hand figure of the Diff. must be placed in the seventh place of decimals and the requisite number of cyphers prefixed. Thus

Diff. = 9 means that the difference is $·0000009$,
Diff. = 74 means that the difference is $·0000074$,
Diff. = 735 means that the difference is $·0000735$,
Diff. = 2021 means that the difference is $·0002021$,
whilst Diff. = 12348 means that the difference is $·0012348$.

161. Page No. 126 also gives the tabular logs. of ratios between 57° and 58°. Suppose we wanted $L \tan 57° 20'$. We now start with the line at the *bottom* of the page and run our eye *up* the column which has Tang. at its *foot*. We go up this column until we arrive at the number which is on the same level as the number 20 in the extreme *right-hand* column. This number we find to be $10·1930286$, which is therefore the value of

$$L \tan 57° 20'.$$

LOGARITHMIC SINES, TANGENTS, SECANTS, ETC.

32 Deg.

′	Sine	Diff.	Cosec.	Tang.	Diff.	Cotang.	Secant	Diff.	Cosine	
0	9·7242097	2021	10·2757903	9·7957892	2811	10·2042108	10·0715795	790	9·9284205	60
1	9·7244118	2020	10·2755882	9·7960703	2810	10·2039297	10·0716585	790	9·9283415	59
2	9·7246138	2018	10·2753862	9·7963513	2809	10·2036487	10·0717375	791	9·9282625	58
3	9·7248156	2018	10·2751844	9·7966322	2808	10·2033678	10·0718166	791	9·9281834	57
4	9·7250174	2015	10·2749826	9·7969130	2808	10·2030870	10·0718957	792	9·9281043	56
5	9·7252189	2015	10·2747811	9·7971938	2807	10·2028062	10·0719749	792	9·9280251	55
6	9·7254204	2013	10·2745796	9·7974745	2806	10·2025255	10·0720541	793	9·9279459	54
7	9·7256217	2012	10·2743783	9·7977551	2805	10·2022449	10·0721334	793	9·9278666	53
8	9·7258229	2011	10·2741771	9·7980356	2804	10·2019644	10·0722127	794	9·9277873	52
9	9·7260240	2009	10·2739760	9·7983160	2804	10·2016840	10·0722921	794	9·9277079	51
10	9·7262249	2008	10·2737751	9·7985964	2803	10·2014036	10·0723715	795	9·9276285	50
11	9·7264257	2007	10·2735743	9·7988767	2802	10·2011233	10·0724510	795	9·9275490	49
12	9·7266264	2005	10·2733736	9·7991569	2801	10·2008431	10·0725305	796	9·9274695	48
13	9·7268269	2004	10·2731731	9·7994370	2800	10·2005630	10·0726101	796	9·9273899	47
14	9·7270273	2003	10·2729727	9·7997170	2800	10·2002830	10·0726897	797	9·9273103	46
15	9·7272276	2002	10·2727724	9·7999970	2799	10·2000030	10·0727694	797	9·9272306	45
16	9·7274278	2000	10·2725722	9·8002769	2798	10·1997231	10·0728491	798	9·9271509	44
17	9·7276278	1999	10·2723722	9·8005567	2798	10·1994433	10·0729289	798	9·9270711	43
18	9·7278277	1998	10·2721723	9·8008365	2797	10·1991635	10·0730087	799	9·9269913	42
19	9·7280275	1998	10·2719725	9·8011161	2796	10·1988839	10·0730886	799	9·9269114	41
20	9·7282271	1996	10·2717729	9·8013957	2796	10·1986043	10·0731686	800	9·9268314	40
21	9·7284267	1993	10·2715733	9·8016752	2795	10·1983248	10·0732486	800	9·9267514	39
22	9·7286260	1993	10·2713740	9·8019546	2794	10·1980454	10·0733286	801	9·9266714	38
23	9·7288253	1991	10·2711747	9·8022340	2794	10·1977660	10·0734087	801	9·9265913	37
24	9·7290244	1990	10·2709756	9·8025133	2793	10·1974867	10·0734888	802	9·9265112	36
25	9·7292234	1989	10·2707766	9·8027925	2792	10·1972075	10·0735690	803	9·9264310	35
26	9·7294223	1988	10·2705777	9·8030716	2791	10·1969284	10·0736493	803	9·9263507	34
27	9·7296211	1988	10·2703789	9·8033506	2790	10·1966494	10·0737296	803	9·9262704	33
28	9·7298197	1985	10·2701803	9·8036296	2790	10·1963704	10·0738099	805	9·9261901	32
29	9·7300182	1983	10·2699818	9·8039085	2789	10·1960915	10·0738904	804	9·9261096	31
30	9·7302165	1983	10·2697835	9·8041873	2788	10·1958127	10·0739708	805	9·9260292	30
31	9·7304148	1981	10·2695852	9·8044661	2786	10·1955339	10·0740513	806	9·9259487	29
32	9·7306129	1980	10·2693871	9·8047447	2786	10·1952553	10·0741319	806	9·9258681	28
33	9·7308109	1978	10·2691891	9·8050233	2786	10·1949767	10·0742125	806	9·9257875	27
34	9·7310087	1977	10·2689913	9·8053019	2784	10·1946981	10·0742931	808	9·9257069	26
35	9·7312064	1976	10·2687936	9·8055803	2784	10·1944197	10·0743739	807	9·9256261	25
36	9·7314040	1975	10·2685960	9·8058587	2783	10·1941413	10·0744546	808	9·9255454	24
37	9·7316015	1974	10·2683985	9·8061370	2782	10·1938630	10·0745354	809	9·9254646	23
38	9·7317989	1972	10·2682011	9·8064152	2781	10·1935848	10·0746163	809	9·9253837	22
39	9·7319961	1971	10·2680039	9·8066933	2781	10·1933067	10·0746972	810	9·9253028	21
40	9·7321932	1970	10·2678068	9·8069714	2780	10·1930288	10·0747782	810	9·9252218	20
41	9·7323902	1968	10·2676098	9·8072494	2779	10·1927506	10·0748592	811	9·9251408	19
42	9·7325870	1967	10·2674130	9·8075273	2779	10·1924727	10·0749403	811	9·9250597	18
43	9·7327837	1966	10·2672163	9·8078052	2777	10·1921948	10·0750214	812	9·9249786	17
44	9·7329803	1965	10·2670197	9·8080829	2777	10·1919171	10·0751026	813	9·9248974	16
45	9·7331768	1963	10·2668233	9·8083606	2777	10·1916394	10·0751839	812	9·9248161	15
46	9·7333731	1962	10·2666269	9·8086383	2775	10·1913617	10·0752651	814	9·9247349	14
47	9·7335693	1961	10·2664307	9·8089158	2775	10·1910842	10·0753465	814	9·9246535	13
48	9·7337654	1960	10·2662346	9·8091933	2774	10·1908067	10·0754279	814	9·9245721	12
49	9·7339614	1958	10·2660386	9·8094707	2773	10·1905293	10·0755093	815	9·9244907	11
50	9·7341572	1957	10·2658428	9·8097480	2773	10·1902520	10·0755908	815	9·9244092	10
51	9·7343529	1956	10·2656471	9·8100253	2772	10·1899747	10·0756723	816	9·9243277	9
52	9·7345485	1955	10·2654515	9·8103025	2771	10·1896975	10·0757539	817	9·9242461	8
53	9·7347440	1953	10·2652560	9·8105796	2770	10·1894204	10·0758356	817	9·9241644	7
54	9·7349393	1952	10·2650607	9·8108566	2770	10·1891434	10·0759173	817	9·9240827	6
55	9·7351345	1951	10·2648655	9·8111336	2769	10·1888664	10·0759990	819	9·9240010	5
56	9·7353296	1950	10·2646704	9·8114105	2768	10·1885895	10·0760809	818	9·9239191	4
57	9·7355246	1949	10·2644754	9·8116873	2768	10·1883127	10·0761627	819	9·9238373	3
58	9·7357195	1947	10·2642805	9·8119641	2767	10·1880359	10·0762446	820	9·9237554	2
59	9·7359142	1946	10·2640858	9·8122408	2766	10·1877592	10·0763266	820	9·9236734	1
60	9·7361088		10·2638913	9·8125174		10·1874826	10·0764086		9·9235914	0
′	Cosine	Diff.	Secant	Cotang.	Diff.	Tang.	Cosec.	Diff.	Sine	′

57 Deg.

EXAMPLES XXV.

1. Find θ, given that $\cos\theta = \cdot 9725382$,
$\cos 13° 27' = \cdot 9725733$, diff. for $1' = 677$.

2. Find the angle whose sine is $\frac{3}{8}$ given.
$\sin 22° 1' = \cdot 3748763$, diff. for $1' = 2696$.

3. Given $\operatorname{cosec} 65° 24' = 1 \cdot 0998243$,
diff. for $1' = 1464$,
find the value of $\operatorname{cosec} 65° 24' 37''$,
and the angle whose cosec is $1 \cdot 0997938$.

4. Given $L \tan 22° 37' = 9 \cdot 6197205$,
diff. for $1' = 3557$,
find the value of $L \tan 22° 37' 22''$,
and the angle whose L tan is $9 \cdot 6195283$.

5. Find the angle whose L cos is $9 \cdot 993$, given
$L \cos 10° 15' = 9 \cdot 9930131$, diff, for $1' = 229$.

6. Find the angle whose L sec is $10 \cdot 15$, given
$L \sec 44° 55' = 10 \cdot 1498843$, diff. for $1' = 1260$.

7. From the table on page 126 find the values of
(1) $L \sin 32° 18' 23''$ (2) $L \cos 32° 16' 49''$
(3) $L \cot 32° 29' 43''$ (4) $L \sec 32° 52' 27''$
(5) $L \tan 57° 45' 28''$ (6) $L \operatorname{cosec} 57° 48' 21''$
and (7) $L \cos 57° 58' 29''$

8. With the help of the same page solve the equations
(1) $L \tan\theta = 10 \cdot 1959261$, (2) $L \operatorname{cosec}\theta = 10 \cdot 0738125$,
(3) $L \cos\theta = 9 \cdot 9259283$ and (4) $L \sin\theta = 9 \cdot 9241352$.

9. Take out of the tables $L \tan 16° 6' 23''$ and calculate the value of the square root of the tangent.

10. Change into a form more convenient for logarithmic computation (*i.e.* express in the form of products of quantities) the quantities.
(1) $1 + \tan x \tan y$, (2) $1 - \tan x \tan y$,
(3) $\cot x + \tan y$, (4) $\cot x - \tan y$,
(5) $\dfrac{1 - \cos 2x}{1 + \cos 2x}$, and (6) $\dfrac{\tan x + \tan y}{\cot x + \cot y}$.

CHAPTER XII

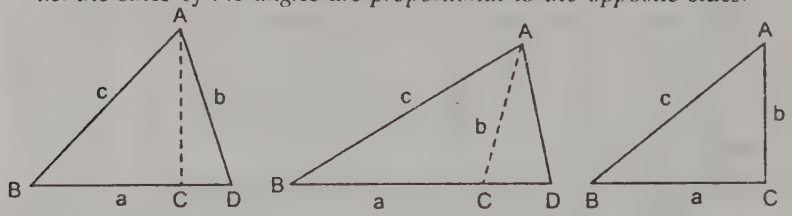
RELATIONS BETWEEN THE SIDES AND THE TRIGONOMETRICAL RATIOS OF THE ANGLES OF ANY TRIANGLE

162. In any triangle ABC, the side BC, opposite to the angle A, is denoted by a; the sides CA and AB, opposite to the angles B and C respectively, are denoted by b and c

163. Theorem. *In any triangle ABC,*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

i.e. the sines of the angles are proportional to the opposite sides.



Draw AD perpendicular to the opposite side meeting it, produced if necessary, in the point D.

In the triangle ABD, we have

$$\frac{AD}{AB} = \sin B, \text{ so that } AD = c \sin B.$$

In the triangle ACD, we have

$$\frac{AD}{AC} = \sin C, \text{ so that } AD = b \sin C.$$

[If the angle C be obtuse, as in the second figure, we have

$$\frac{AD}{b} = \sin ACD = \sin (180° - C) = \sin C \qquad \text{(Art. 72)}$$

so that $AD = b \sin C$.]

Equating these two values of AD, we have

$$c \sin B = b \sin C,$$

i.e.

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

In a similar manner, by drawing a perpendicular from B upon CA, we have

$$\frac{\sin C}{c} = \frac{\sin A}{a}.$$

If one of the angles, C, be a right angle, as in the third figure, we have

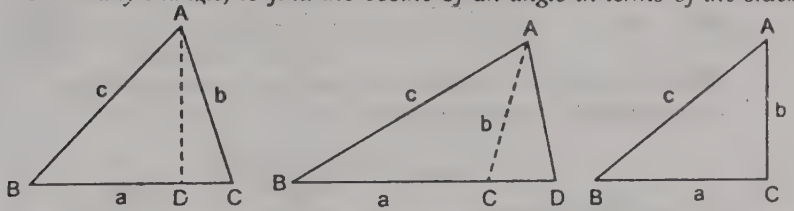
$$\sin C = 1, \sin A = \frac{a}{c}, \text{ and } \sin B = \frac{b}{c}.$$

Hence $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{1}{c} = \frac{\sin C}{c}.$$

We therefore have, in all cases,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

164. *In any triangle, to find the cosine of an angle in terms of the sides.*



Let ABC be the triangle and let the perpendicular from A on BC meet it, produced if necessary, in the point D,

First, let the angle C be **acute,** as in the first figure, By Geometry, we have

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD \qquad \text{...(i)}$$

But $\frac{CD}{CA} = \cos C$, so that $CD = b \cos C$.

Hence (i) becomes

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

i.e. $$2ab \cos C = a^2 + b^2 - c^2,$$

i.e. $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Secondly, let the angle C be **obtuse,** as in the second figure.

By Geometry, we have

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD \qquad \text{...(ii)}$$

But $$\frac{CD}{CA} = \cos ACD = \cos (180° - C) = -\cos C \quad \text{(Art. 72)}$$

So that $$CD = -b \cos C.$$

Hence (ii) becomes

$$c^2 = a^2 + b^2 + 2a(-b \cos C) = a^2 + b^2 - 2ab \cos C$$

so that, as in the first case, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

In a similar manner it may be shewn that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

and $$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

If one of the angle, C be a right angle, the above formula would give $c^2 = a^2 + b^2$, so that $\cos C = 0$. This is correct, since C is a right angle.

The above formula is therefore true for all values of C.

Ex. If $a = 15, b = 36$, and $c = 39$,

then $$\cos A = \frac{36^2 + 39^2 - 15^2}{2 \times 36 \times 39} = \frac{3^2 (12^2 + 13^2 - 5^2)}{2 \times 3^2 \times 12 \times 13} = \frac{288}{24 \times 13} = \frac{12}{13}.$$

165. *To find the sines of half the angles in terms of the sides.*

In any triangle we have, by Art. 164,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

By Art. 109, we have

$$\cos A = 1 - 2 \sin^2 \frac{A}{2}.$$

Hence $$2 \sin^2 \frac{A}{2} = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b^2 + c^2 - 2bc)}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{[a + (b - c)][a - (b - c)]}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc} \quad ...(1)$$

Let $2s$ stand for $a + b + c$, so that s is equal to half the sum of the sides of the triangle, *i.e.* s is equal to the semi-perimeter of the triangle.

We then have

$$a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c),$$

and $$a - b + c = a + b + c - 2b = 2s - 2b = 2(s - b).$$

The relation (1) therefore becomes

$$2 \sin^2 \frac{A}{2} = \frac{2(s - c) \times 2(s - b)}{2bc} = 2 \frac{(s - b)(s - c)}{bc}$$

$$\therefore \quad \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}} \quad ...(2)$$

Similarly,

$$\sin \frac{B}{2} = \sqrt{\frac{(s - c)(s - a)}{ca}}, \text{ and } \sin \frac{C}{2} = \sqrt{\frac{(s - a)(s - b)}{ab}}.$$

166. *To find the cosines of half the angles in terms of the sides.*
By Art. 109, we have

$$\cos A = 2\cos^2\frac{A}{2} - 1.$$

Hence
$$2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc}$$
$$= \frac{[(b+c)+a][(b+c)-a]}{2bc}$$
$$= \frac{(a+b+c)(b+c-a)}{2bc} \quad \ldots(1)$$

Now $b + c - a = a + b + c - 2a = 2s - 2a = 2(s - a)$, so that (1) becomes

$$2\cos^2\frac{A}{2} = \frac{2s \times 2(s-a)}{2bc} = 2\frac{s(s-a)}{bc}.$$

$$\therefore \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \quad \ldots(2)$$

Similarly,

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}, \text{ and } \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}.$$

167. *To find the tangents of half the angles in terms of the sides.*

since
$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}},$$

we have, by (2) of Arts. 165 and 166,

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \div \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \text{ and } \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Since, in a triangle, A is always $< 180°$, $\frac{A}{2}$ is always $< 90°$.

The sine, cosine, and tangent of $\frac{A}{2}$ are therefore always positive (Art. 52).

The positive sign must therefore always be prefixed to the radical sign in the formulae of this and the previous two articles.

168. Ex. If $a = 13, b = 14$ and $c = 15$,

then $$s = \frac{13 + 14 + 15}{2} = 21, \; s - a = 8, \; s - b = 7,$$

and $$s - c = 6$$

Hence $$\sin\frac{A}{2} = \sqrt{\frac{7 \times 6}{14 \times 15}} = \frac{1}{\sqrt{5}} = \frac{1}{5}\sqrt{5},$$

$$\sin\frac{B}{2} = \sqrt{\frac{6 \times 8}{15 \times 13}} = \frac{4}{\sqrt{65}} = \frac{4}{65}\sqrt{65},$$

$$\cos\frac{C}{2} = \sqrt{\frac{21 \times 6}{13 \times 14}} = \frac{3}{\sqrt{13}} = \frac{3}{13}\sqrt{13},$$

and $$\tan\frac{B}{2} = \sqrt{\frac{6 \times 8}{21 \times 7}} = \frac{4}{7}.$$

169. *To express the sine of any angle of a triangle in terms of the sides.*
We have, by Art. 109,

$$\sin A = 2 \sin\frac{A}{2}\cos\frac{A}{2}$$

But, by the previous articles,

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ and } \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

Hence $$\sin A = 2\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}$$

$\therefore$ $$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

EXAMPLES. XXVI.

In a triangle

1. Given $a = 25, b = 52$, and $c = 63$,
   find $\tan\frac{A}{2}, \tan\frac{B}{2}$, and $\tan\frac{C}{2}$.
2. Given $a = 125, b = 123$, and $c = 62$.
   find the sines of half the anlges and the sines of the angles.
3. Given $a = 18, b = 24$ and $c = 30$
   find $\sin A, \sin B$, and $\sin C$.
4. Given $a = 35, b = 84$ and $c = 91$,
   find $\tan A, \tan B$, and $\tan C$
5. Given $a = 13, b = 14$, and $c = 15$,
   find the sines of the angles. Verify by a graph.
6. Given $a = 287, b = 816$, and $c = 865$,
   find the values of $\tan\frac{A}{2}$ and $\tan A$.

7. Given $a = \sqrt{3}, b = \sqrt{2}$, and $c = \dfrac{\sqrt{6} + \sqrt{2}}{2}$.

find the angles.

170. *In any triangle, to prove that*

$$a = b \cos C + c \cos B.$$

Take the figures of Art. 164.

In the first case, we have

$$\frac{BD}{BA} = \cos B, \text{ so that } BD = c \cos B,$$

and

$$\frac{CD}{CA} = \cos C, \text{ so that } CD = b \cos C.$$

Hence $a = BC = BD + DC = c \cos B + b \cos C$.

In the second case, we have

$$\frac{BD}{BA} = \cos B, \text{ so that } BD = c \cos B,$$

and

$$\frac{CD}{CA} = \cos ACD = \cos (180° - C)$$

$$= -\cos C \text{ (Art. 72)},$$

so that $CD = -b \cos C$.

Hence, in this case,

$$a = BC = BD - CD = c \cos B - (-b \cos C),$$

so that in each case

$$a = b \cos C + c \cos B.$$

Similarly, $b = c \cos A + a \cos C$,

and $c = a \cos B + b \cos A$.

171. *In any triangle, to prove that*

$$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}.$$

In any triangle, we have $\dfrac{b}{c} = \dfrac{\sin B}{\sin C}$.

$$\frac{b - c}{b + c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2 \cos \dfrac{B + C}{2} \sin \dfrac{B - C}{2}}{2 \sin \dfrac{B + C}{2} \cos \dfrac{B - C}{2}}$$

$$= \frac{\tan \dfrac{B - C}{2}}{\tan \dfrac{B + C}{2}} = \frac{\tan \dfrac{B - C}{2}}{\tan \left(90° - \dfrac{A}{2}\right)}$$

$$= \frac{\tan \dfrac{B - C}{2}}{\cot \dfrac{A}{2}} \text{ (Art. 69)}.$$

Hence $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$.

172. Ex. *From the formulae of Art. 164 deduce those of Art. 170 and vice versa.*

The first and third formulae of Art. 164 give

$$b\cos C + c\cos B = \frac{a^2+b^2-c^2}{2a} + \frac{c^2+a^2-b^2}{2a}$$

$$= \frac{2a^2}{2a} = a,$$

so that $a = b\cos C + c\cos B$.

Similarly, the other formulae of Art. 170 may be obtained

Again, the three formulae of Art. 170 give

$$a = b\cos C + c\cos B,$$

$$b = c\cos A + a\cos C,$$

and $$c = a\cos B + b\cos A.$$

Multiplying these in succession by a, b and $-c$, we have, by addition, $a^2+b^2-c^2 = a(b\cos C + c\cos B) + b(c\cos A + a\cos C) - c(a\cos B + b\cos A) = 2ab\cos C$.

$$\therefore \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Similarly, the other formulae of Art. 162 may be found.

173. *The student will often meet with identities, which he is required to prove, which involve both the sides and the angles of a triangle.*

It is, in general, desirable in the identity to substitute for the sides in terms of the angles, or to substitute for the ratios of the angles in terms of the sides.

Ex. 1. *Prove that* $a\cos\frac{B-C}{2} = (b+c)\sin\frac{A}{2}$.

By Art. 163, we have

$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \frac{\cos\frac{A}{2}\cos\frac{B-C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}.$$

$$\therefore \quad (b+c)\sin\frac{A}{2} = a\cos\frac{B-C}{2}.$$

Ex. 2. *In any triangle prove that*

$$(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C = 0.$$

By Art. 163 we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}.$$

Hence the given expression

$$= (b^2 - c^2)\frac{\cos A}{ak} + (c^2 - a^2)\frac{\cos B}{bk} + (a^2 - b^2)\frac{\cos C}{ck}$$

$$= \frac{1}{k}\left[(b^2 - c^2)\frac{b^2 + c^2 - a^2}{2abc} + (c^2 - a^2)\frac{c^2 + a^2 - b^2}{2abc} + (a^2 - b^2)\frac{a^2 + b^2 - c^2}{2abc}\right]$$

$$= \frac{1}{2abck}[b^4 - c^4 - a^2(b^2 - c^2) + c^4 - a^4 - b^2(c^2 - a^2) + a^4 - b^4 - c^2(a^2 - b^2)]$$

$$= 0,$$

Ex. 3. *In any triangle prove that*

$$(a + b + c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}.$$

The left-hand member

$$= 2s\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\right] \text{ by Art. 167}$$

$$= 2s\sqrt{\frac{s-c}{s}}\left[\sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}}\right]$$

$$= 2\sqrt{s(s-c)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right]$$

$$= \frac{2\sqrt{s(s-c)}\cdot c}{\sqrt{(s-a)(s-b)}}, \text{ since } 2s = a + b + c,$$

$$= 2c\cot\frac{C}{2}.$$

This identity may also be proved by substituting for the sides.

We have, by Art. 163,

$$\frac{a+b+c}{c} = \frac{\sin A + \sin B + \sin C}{\sin C}$$

$$= \frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}, \text{ as in Art. 127, } = \frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}},$$

Also $$\frac{2\cot\frac{C}{2}}{\tan\frac{A}{2} + \tan\frac{B}{2}} = \frac{2\cos\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}\left[\sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}\right]}$$

$$= \frac{2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{C}{2}\sin\frac{A+B}{2}} = \frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}. \qquad \text{(Art. 69.)}$$

We have therefore

$$\frac{a+b+c}{c} = \frac{2\cot\frac{C}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}},$$

so that $$(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}.$$

Ex. 4. *If the sides of a triangle be in Arithmetical Progression, prove that so also are the cotangents of half the angles.*

We have given that

$$a + c = 2b \qquad \ldots(1)$$

and we have to prove that

$$\cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2} \qquad \ldots(2)$$

Now (2) is true if

$$\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 2\sqrt{\frac{s(s-b)}{(s-c)(s-a)}},$$

or, by multiplying both sides by

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

if $(s-a)+(s-c) = 2(s-b)$,

i.e. if $2s-(a+c) = 2s-2b$,

i.e. if $a + c = 2b$, which is relation (1).

Hence if relation (1) be true, so also is relation (2).

EXAMPLES. XXVII.

In any triangle ABC, prove that

1. $\sin\frac{B-C}{2} = \frac{b-c}{a}\cos\frac{A}{2}$.
2. $b^2\sin 2C + c^2\sin 2B = 2bc\sin A$.
3. $a(b\cos C - c\cos B) = b^2 - c^2$.
4. $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$.
5. $a(\cos B + \cos C) = 2(b+c)\sin^2\frac{A}{2}$.
6. $a(\cos C - \cos B) = 2(b-c)\cos^2\frac{A}{2}$.

7. $\dfrac{\sin(B-C)}{\sin(B+C)} = \dfrac{b^2-c^2}{a^2}$.

8. $\dfrac{a+b}{a-b} = \tan\dfrac{A+B}{2}\cot\dfrac{A-B}{2}$.

9. $a\sin\left(\dfrac{A}{2}+B\right) = (b+c)\sin\dfrac{A}{2}$.

In any triangle ABC, prove that

10. $\dfrac{a^2\sin(B-C)}{\sin B+\sin C} + \dfrac{b^2\sin(C-A)}{\sin C+\sin A} + \dfrac{c^2\sin(A-B)}{\sin A+\sin B} = 0$.

11. $(b+c-a)\left(\cot\dfrac{B}{2}+\cot\dfrac{C}{2}\right) = 2a\cot\dfrac{A}{2}$.

12. $a^2+b^2+c^2 = 2(bc\cos A + ca\cos B + ab\cos C)$.

13. $(a^2-b^2+c^2)\tan B = (a^2+b^2-c^2)\tan C$.

14. $c^2 = (a-b)^2\cos^2\dfrac{C}{2} + (a+b)^2\sin^2\dfrac{C}{2}$.

15. $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$.

16. $\dfrac{a\sin(B-C)}{b^2-c^2} = \dfrac{b\sin(C-A)}{c^2-a^2} = \dfrac{c\sin(A-B)}{a^2-b^2}$.

17. $a\sin\dfrac{A}{2}\sin\dfrac{B-C}{2} + b\sin\dfrac{B}{2}\sin\dfrac{C-A}{2} + c\sin\dfrac{C}{2}\sin\dfrac{A-B}{2} = 0$.

18. $a^2(\cos^2 B-\cos^2 C) + b^2(\cos^2 C-\cos^2 A) + c^2(\cos^2 A-\cos^2 B) = 0$.

19. $\dfrac{b^2-c^2}{a^2}\sin 2A + \dfrac{c^2-a^2}{b^2}\sin 2B + \dfrac{a^2-b^2}{c^2}\sin 2C = 0$.

20. $\dfrac{(a+b+c)^2}{a^2+b^2+c^2} = \dfrac{\cot\dfrac{A}{2}+\cot\dfrac{B}{2}+\cot\dfrac{C}{2}}{\cot A+\cot B+\cot C}$.

21. $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) = 3abc$.

22. In a triangle whose sides are 3, 4, and $\sqrt{38}$ cm respectively, prove that the largest angle is greater than 120°.

23. The sides of a right-angled triangle are 21 and 28 cm; find the length of the perpendicular drawn to the hypothenuse from the right angle.

24. If in any triangle the angles be to one another as 1 : 2 : 3, prove that the corresponding sides are as $1 : \sqrt{3} : 2$.

25. In any triangle, if

$$\tan\frac{A}{2} = \frac{5}{6} \text{ and } \tan\frac{B}{2} = \frac{20}{37},$$

find $\tan\dfrac{C}{2}$, and prove that in this triangle $a + c = 2b$

26. In an isosceles right-angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Shew that it divides the angle into parts whose contangents are 2 and 3.

27. The perpendicular AD to the base of a triangle ABC divides it into segments such that BD, CD and AD are in the ratio of 2, 3 and 6; prove that the vertical angle of the triangle is 45°.

28. A ring, ten cm in diameter, is suspended from a point 12 cm above its centre by 6 equal strings attached to its circumference at equal intervals. Find the cosine of the angle between consecutive strings.

29. If a^2, b^2 and c^2 be in A.P., prove that $\cot A$, $\cot B$ and $\cot C$ are in A.P. also.

30. If a, b, and c be in A.P., prove that $\cos A \cot \frac{A}{2}$, $\cos B \cot \frac{B}{2}$, and $\cos C \cot \frac{C}{2}$ are in A.P.

31. If a, b, and c are in H.P. prove that $\sin^2 \frac{A}{2}$, $\sin^2 \frac{B}{2}$ and $\sin^2 \frac{C}{2}$ are also in H.P.

32. The sides of a triangle are in A.P. and the greatest and least angles are θ and ϕ, prove that
$$4(1-\cos\theta)(1-\cos\phi) = \cos\theta + \cos\phi.$$

33. The sides of a triangle are in A.P. and the greatest angles exceeds the least by 90°; prove that the sides are proportional to $\sqrt{7}+1$, $\sqrt{7}$, and $\sqrt{7}-1$.

34. If $C = 60°$, then prove that
$$\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}.$$

35. In any triangle ABC if D be any point of the base BC, such that $BD : DC :: m : n$, and if $\angle BAD = \alpha$, $\angle DAC = \beta$, $\angle CDA = \theta$, and $AD = x$, prove that
$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta = n\cot B - m\cot C,$$
and
$$(m+n)^2 \cdot x^2 = (m+n)(mb^2 + nc^2) - mn\alpha^2.$$

36. If in a triangle the bisector of the side c be perpendicular to the side b, prove that
$$2\tan A + \tan C = 0.$$

37. In any triangle prove that, if θ be any angle, then
$$b\cos\theta = c\cos(A-\theta) + a\cos(C+\theta).$$

38. If p and q be the perpendiculars from the angular points A and B on any line passing through the vertex C of the triangle ABC, then prove that
$$a^2p^2 + b^2q^2 - 2abpq\cos C = a^2b^2\sin^2 C.$$

39. In the triangle ABC, line OA; OB, and OC are drawn so that the angles OAB, OBC and OCA are each equal to ω; prove that
$$\cot\omega = \cot A + \cot B + \cot C,$$
and
$$\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$$

CHAPTER XIII

SOLUTION OF TRIANGLES

174. In any triangle the three sides and the three angles are often called the elements of the triangle. When any three elements of the triangle are given, provided they be not the three angles, the triangle is in general completely known, *i.e.* its other angles and sides can be calculated. When the three angles are given, only the ratios of the lengths of the sides can be found, so that the triangle is given in *shape* only and not in size. When three elements of a triangle are given, the process of calculating its other three elements is called the **Solution of the Triangle.**

We shall first discuss the solution of right-angled triangles *i.e.* triangles which have one angle given equal to a right angle.

The next four articles refer to such triangles, and C denotes the right angle.

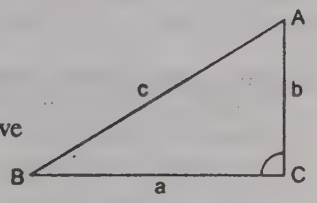
175. Case I : *Given the hypothenuse and one side, to solve the triangle.*

Let b be the given side and c the given hypothenuse.

The angle B is given by the relation

$$\sin B = \frac{b}{c}.$$

$$\therefore L \sin B = 10 + \log b - \log c.$$



Since b and c are known, we thus have $L \sin B$ and therefore B.

The angle $A\ (= 90° - B)$ is then known.

The side a is obtained from either of the relations

$$\cos B = \frac{a}{c}, \tan B = \frac{b}{a}, \text{ or } a = \sqrt{(c-b)(c+b)}$$

176. Case II : *Given the two sides a and b, to solve the triangle.*

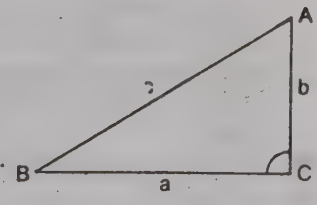
Here B is given by

$$\tan B = \frac{b}{a}$$

so that $L \tan B = 10 + \log b = \log a.$

Hence $L \tan B$, and therefore B, is known.

The angle $A\ (= 90° - B)$ is then known.



The hypothenuse c is given by the relation $c = \sqrt{a^2 + b^2}$.

This relation is not however very suitable for logarithmic calculation, and c is best given be

$$\sin B = \frac{b}{c} \text{ i.e. } c = \frac{b}{\sin B}.$$

$$\therefore \quad \log c = \log b - \log \sin B$$
$$= 10 + \log b - L \sin B.$$

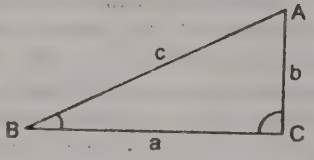
Hence c is obtained.

177. Case III : *Given an angle B and one of the sides a, to solve the triangle.*

Here $A\ (= 90° - B)$ is known

The side b is found from the relation

$$\frac{b}{a} = \tan B,$$



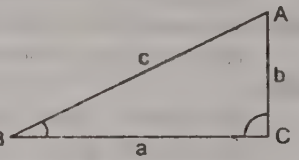
and c from the relation.

$$\frac{a}{c} = \cos B.$$

178. Case IV : *Given an angle B and the hypothenuse c to solve the triangle.*

Here A is known, and a and b are obtained from the relations

$$\frac{a}{c} = \cos B, \text{ and } \frac{b}{c} = \sin B.$$



EXAMPLES. XXVIII.

1. In a right-angled triangle ABC, where C is the right angle, if $a = 50$ and $B = 75°$, find the sides. $(\tan 75° = 2 + \sqrt{3}.)$
2. Solve the triangle of which two sides are equal to 10 and 20 cm and of which the included angle is 90°; given that $\log 20 = 1.30103$, and
   $L \tan 26° 33' = 9.6986847$, diff. for $1' = 3160$.
3. The length of the perpendicular from one angle of a triangle upon the base is 3 inches and the lengths of the sides containing this angle are 4 and 5 cm. Find the angles, having given
   $\log 2 = .30103, \log 3 = .4771213,$
   $L \sin 36° 52' = 9.7781186$, diff. for $1' = 1684$,
   and $L \sin 48° 35' = 9.8750142$, diff, for $1' = 1115$.
4. Find the acute angles of a right-angled triangle whose hypothenuse is four times as long as the perpendicular drawn to it from the opposite angle.

179. We now proceed to the case of the triangle which is not given to be right-angled.

The different cases to be considered are :

Case I. *The three sides given;*

Case II. *Two sides and the included angle given;*

Case III. *Two sides and the angle opposite one of them given;*

Case IV. *One side and two angles given;*

Case V. *The three angles given.*

180. Case I. *The three sides a, b, and c given.*

Since the sides are known, the semi-perimeter s is known and hence also the quantities $s-a, s-b$, and $s-c$.

The half-angles $\frac{A}{2}, \frac{B}{2}$, and $\frac{C}{2}$ are then found from the formulae

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \qquad \tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

and $$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Only two of the angles need be found, the third being known since the sum of the three angles is always 180°.

The angles may also be found by using the formulae for the sine or cosine of the semi-angles. (Arts. 165 and 166.)

The above formulae are all suited for logarithmic computation.

The angle A may also be obtained from the formula

$$\cos A=\frac{b^2+c^2-a^2}{2bc}. \qquad \text{(Art. 164.)}$$

This formula is not, in general, suitable for logarithmic calculation. it may be conveniently used however when the sides, a, b, and c are small numbers.

Ex. *The sides of a triangle are 32, 40 and 66; find the angle opposite the greatest side, by using the five-figure tables at the end of this book.*

Here $a=32, b=40$ and $c=66$,

so that $$s=\frac{32+40+66}{2}=69, s-a=37, s-b=29$$

and $$s-c=3$$

Hence $$\cot\frac{C}{2}=\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=\sqrt{\frac{69\times 3}{37\times 29}}.$$

$\therefore$ $$L\cot\frac{C}{2}=10+\frac{1}{2}[\log 3+\log 69-\log 37-\log 29]$$

$$=10+\frac{1}{2}[.47712+1.83885-1.56820-1.46240]$$

$$=10+1.157985-1.51530$$

$$=9.642685 \qquad \ldots(1)$$

From the Logarithmic Cotangents table, we have

$$L\cot 66°\,10'=9.64517 \qquad \ldots(2)$$

and $$L\cot 66°\,20'=9.64175 \qquad \ldots(3)$$

and hence we see that $\frac{C}{2}$ lies between 66° 10′ and 66° 20′

Let then $\frac{C}{2} = 66° 10' + x'$.

For an *increase* of x' in the angle we see, from (1) and (2), that the *decrease* in the logarithm = .002485.

For an *increase* of 10′ in the angle we see, from (2) and (3), that the *decrease* in the logarithm = .00342

Hence $\frac{x}{10} = \frac{.002485}{.00342} = \frac{2485}{3420}$.

$\therefore$ $x' = 7\frac{91'}{342} = 7' 16''$ nearly

$\therefore \frac{C}{2} = 66° 17' 16''$, and hence $C = 132° 34' 32''$

EXAMPLES. XXIX.

[The student should verify the results of some of the following examples (e.g. Nos. 1, 7, 8, 10, 11, 12) by an accurate graph.]

1. If the sides of a triangle be 56, 65, and 33 cm, find the greatest angle.
2. The sides of a triangle are 7, $4\sqrt{3}$, and $\sqrt{13}$ metres respectively. Find the number of degrees in its smallest angle.
3. The sides of a triangle are $x^2 + x + 1$, $2x + 1$, and $x^2 - 1$; prove that the greatest angle is 120°.
4. The sides of a triangle are a, b and $\sqrt{a^2 + ab + b^2}$ cm; find the greatest angle.
5. If $a = 2, b = \sqrt{6}$, and $c = \sqrt{3} - 1$, solve the triangle.
6. If $a = 2, b = \sqrt{6}$, and $c = \sqrt{3} + 1$, solve the triangle.
7. If $a = 9, b = 10$, and $c = 11$, find B, given
   $\log 2 = .30103$, $L \tan 29° 29' = 9.7523472$,
   and $L \tan 29° 30' = 9.7526420$.
8. The sides of a triangle are 130, 123, and 77 cm. Find the greatest angle, having given
   $\log 2 = .30103$, $L \tan 38° 39' = 9.9029376$,
   and $L \tan 38° 40' = 0.9031966$.
9. Find the greatest angle of a triangle whose sides are 242, 188, and 270 feet, having given
   $\log 2 = .30103$, $\log 3 = .4771213$, $\log 7 = .8450980$,
   $L \tan 38° 20' = 9.8980104$, and $L \tan 38° 19' = 9.8977507$,
10. The sides of a triangle are 2, 3 and 4; find the greatest angle, having given
    $\log 2 = .30103$, $\log 3 = .4771213$,
    $L \tan 52° 14' = 10.1108395$,
    and $L \tan 52° 15' = 10.1111004$.

Making use of the tables, find all the angles when

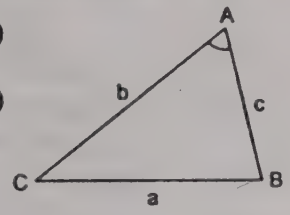
11. $a = 25, b = 26$, and $c = 27$.
12. $a = 17, b = 20$, and $c = 27$.
13. $a = 2000, b = 1050$, and $c = 1150$.

181. Case II. *Given two sides b and c and the included angle A.*

Taking b to be the greater of the two given sides, we have

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} \quad \text{(Art. 171)} \qquad ...(1)$$

and
$$\frac{B+C}{2} = 90° - \frac{A}{2} \qquad ...(2)$$



These two relations give us

$$\frac{B-C}{2} \text{ and } \frac{B+C}{2},$$

and therefore, by addition and subtraction, B and C.

The third side a is then known from the relation.

$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

which gives
$$a = b\frac{\sin A}{\sin B},$$

and thus determines a.

The side a may also be found from the formula

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

This is not adapted to logarithmic calculation but is sometimes useful, especially when the sides a and b are small numbers.

182. Ex. 1. If $b = \sqrt{3}$, $c = 1$, and $A = 30°$, solve the triangle.

We have

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{\sqrt{3}-1}{\sqrt{3}+1}\cot 15°.$$

Now
$$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} \text{ (Art. 101)},$$

so that
$$\cot 15° = \frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

Hence
$$\tan\frac{B-C}{2} = 1.$$

$\therefore$
$$\frac{B-C}{2} = 45° \qquad ...(1)$$

Also
$$\frac{B+C}{2} = 90° - \frac{A}{2} = 90° - 15° = 75° \qquad ...(2)$$

By addition, $B = 120°$.

By substraction, $C = 30°$.

Since $A = C$, we have $a = c = 1$.

Otherwise. We have

$$a^2 = b^2 + c^2 - 2bc\cos A = 3 + 1 - 2\sqrt{3}\cdot\frac{\sqrt{3}}{2} = 1,$$

so that $a = 1 = c$.

$\therefore$
$$C = A = 30°,$$

and
$$B = 180° - A - C = 120°.$$

Ex. 2. *If* $b = 215, c = 105,$ *and* $A = 74° 27'$, *find the remaining angles and also the third side a, having given*

$\log 2 = .3010300$, $\log 11 = 1.0413927$,

$\log 105 = 2.0211893$, $\log 212.476 = 2.3273103$,

$L \cot 37° 13' = 10.1194723$, *diff.* for $1' = 2622$,

$L \tan 24° 20' = 9.6553477$, *diff.* for $1' = 3364$,

$L \sin 74° 27' = 9.9838052$,

and $L \operatorname{cosec} 28° 25' = 10.3225025$, *diff.* for $1' = 2334$.

Here $$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{11}{32} \cot 37° 13' 30''$$

Now $L \cot 37° 13' = 10.1194723$

diff. for $30'' = -\quad 1311$

$\therefore L \cot 37° 13' 30'' = 10.1195412$

$\log 11 = \quad 1.0413927$

11.1607339

$\log 32 = 1.50515$

$\therefore L \tan \frac{1}{2}(B-C) = 9.6555839$

But $L \tan 24° 20' = 9.6553477$

diff. $= 2362$

$= \text{diff. for } \frac{2}{3}\frac{3}{3}\frac{6}{6}\frac{2}{4} \text{ of } 60''$

$= \text{diff. for } 42.1''.$

$\therefore \quad \frac{B-C}{2} = 24° 20' 42''.$

But $\quad \frac{B+C}{2} = 90° - \frac{A}{2} = 52' 46' 30''$

$\therefore$ by addition, $B = 77° 7' 12''$

and, by subtraction $C = 28° 25' 48''$

Again $\quad \frac{a}{\sin A} = \frac{c}{\sin C} = c \operatorname{cosec} C,$

$\therefore \quad a = 105 \sin 74° 27' \operatorname{cosec} 28° 25' 48''$

But $L \operatorname{cosec} 28° 25' = 10.3225025$

diff. for $48'' = -\ 1867$

$L \operatorname{cosec} 28° 25' 48'' = 10.3223158$

$L \sin 74° 27' = 9.9838052$

$\log 105 = 2.0211893$

$\cdot 30103$

$\quad 5$

$1 \cdot 50515$

$\quad 2362$

$\quad\quad 60$

$3364 \overline{)\ 141720\ (\ 42 \cdot 1}$

$\quad\quad 13456$

$\quad\quad\ 7160$

$\quad\quad\ 6728$

$\quad\quad\ \ 4320$

$\frac{48}{60} \times 2334$

$= \frac{4}{5} \times 2334$

$= 1867.$

22.3273103
20

$\therefore \log a = 2.3273103$

$\therefore a = 212.476$

***183.** *There are ways of finding the third side a of the triangle in the previous case without first finding the angles B and C.*

Two methods are as follows :

(1) Since $a^2 = b^2 + c^2 - 2bc\cos A$

$$= b^2 + c^2 - 2bc\left(2\cos^2\frac{A}{2} - 1\right)$$

$$= (b + c)^2 = 4bc\cos^2\frac{A}{2}.$$

$$\therefore \quad a^2 = (b + c)^2\left[1 - \frac{4bc}{(b + c)^2}\cos^2\frac{A}{2}\right].$$

Hence if $\sin^2\theta = \dfrac{4bc}{(b + c)^2}\cos^2\dfrac{A}{2},$

we have $a^2 = (b + c)^2[1 - \sin^2\theta] = (b + c)^2\cos^2\theta,$

so that $a = (b + c)\cos\theta$

If then $\sin\theta$ be calculated from the relation:

$$\sin\theta\,\frac{2\sqrt{bc}}{b + c}\cos\frac{A}{2},$$

we have $a = (b + c)\cos\theta.$

(2) We have

$$a^2 = b^2 + c^2 - 2bc\cos A = b^2 + c^2 - 2bc\left(1 - 2\sin^2\frac{A}{2}\right)$$

$$= (b - c)^2 + 4bc\sin^2\frac{A}{2}$$

$$= (b - c)^2\left[1 + \frac{4bc}{(b - c)^2}\sin^2\frac{A}{2}\right].$$

Let $\dfrac{4bc}{(b - c)^2}\sin^2\dfrac{A}{2} = \tan^2\phi,$

so that $\tan\phi = \dfrac{2\sqrt{bc}}{b - c}\sin\dfrac{A}{2},$

and hence ϕ is known.

Then $a^2 = (b - c)^2[1 + \tan^2\phi] = \dfrac{(b - c)^2}{\cos^2\phi},$

so that $a = (b - c)\sec\phi,$

and is therefore easily found.

An angle, such as θ or ϕ above, introduced for the purpose of facilitating calculation is called a subsidiary angle (Art. 129).

EXAMPLES. XXX.

[*The student should verify the results of some of the following examples (e.g. Nos.* 4, 5, 6, 11) *by an accurate graph.*]

1. If b 90, $c = 70$, and $A = 72° 48' 30''$, find B and C, given
$\log 2 = \cdot 30103$, $L \cot 36° 24' 15'' = 10 \cdot 1323111$,
$L \tan 9° 37' = 9 \cdot 2290071$,
and $L \tan 9° 38' = 9 \cdot 2297735$.

2. If $a = 21$, $b = 11$, and $c = 34° 42' 30''$, find A and B, given
$\log 2 = \cdot 30103$,
and $L \tan 72° 38' 45'' = 10 \cdot 50515$.

3. If the angles of a triangle be in A.P. and the lengths of the greatest and least sides be 24 and 16 cm respectively, find the length of the third side and the angles, given
$\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$,
and $L \tan 19° 6' = 9 \cdot 5394287$, diff. for $1' = 4084$.

4. If $a = 13$, $b = 7$, and $C = 60°$, find A and B, given that
$\log 3 = \cdot 4771213$,
and $L \tan 27° 27' = 9 \cdot 7155508$, diff for $1' = 3087$.

5. If $a = 2b$, and $C = 120°$, find the values of A, B, and the ratio of c to a, given that
$\log 3 = \cdot 4771213$,
and $L \tan 10° 53' = 9 \cdot 2839070$, dif. for $1' = 6808$.

6. If $b = 14$, $c = 11$ and $A = 60°$, find B and C, given that
$\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$,
$L \tan 11° 44' = 9 \cdot 3174299$,
and $L \tan 11° 45' = 9 \cdot 3180640$.

7. The two sides of a triangle are 540 and 420 meters long respectively and include an angle of $52° 6'$. Find the remaining angles, given that
$\log 2 = \cdot 30103$, $L \tan 26° 3' = 9 \cdot 6891430$,
$L \tan 14° 20' = 9 \cdot 4074189$,
and $L \tan 14° 21' = 9 \cdot 4079453$.

8. If $b = 2\frac{1}{2}$ cm, $c = 2$ cm, and $A = 22° 20'$, find the other angles, and shew that the third side is nearly one cm, given
$\log 2 = \cdot 30103$, $\log 3 = \cdot 47712$,
$L \cot 11° 10' = 10 \cdot 70465$, $L \sin 22° 20' = 9 \cdot 57977$
$L \tan 29° 22' 20'' = 9 \cdot 75038$, $L \tan 29° 22' 30'' = 9 \cdot 75043$
and $L \sin 49° 27' 34'' = 9 \cdot 88079$.

9. If $a = 2$, $b = 1 + \sqrt{3}$, and $C = 60°$, solve the triangle.

10. Two sides of a triangle are $\sqrt{3} + 1$ and $\sqrt{3} - 1$, and the included angle is 60°; find the other sides and angles.

11. If $b = 1, c = \sqrt{3} - 1$, and $A = 60°$, find the length of the side a.

12. If $b = 91, c = 125$, and $\tan \frac{A}{2} = \frac{17}{6}$, prove that $a = 204$.

13. If $a = 5, b = 4$ and $\cos (A - B) = \frac{31}{32}$, prove that the third side c will be 6.

14. One angle of a triangle is 30° and the lengths of the sides adjacent to it are 40 and $40\sqrt{3}$ metres. Find the length of the third side and the number of degrees in the other angles.

15. The sides of a triangle are 9 and 3, and the difference of the angles opposite to them is 90°. Find the base and the angles, having given

$$\log 2 = \cdot 30103, \log 3 = \cdot 4771213,$$
$$\log 75894 = 4 \cdot 8802074, \log 75896 = 4 \cdot 8802132,$$
$$L \tan 26° 33' = 9 \cdot 6986847,$$
and $$L \tan 26° 34' = 9 \cdot 6990006.$$

16. If $\tan \phi = \frac{a - b}{a + b} \cot \frac{C}{2}$,

prove that $c = (a + b) \frac{\sin \frac{C}{2}}{\cos \phi}$.

If $a = 3, b = 1$, and $C = 53° 7' 48''$, find c without getting A and B, given

$$\log 2 = \cdot 30103, \log 25298 = 4 \cdot 4030862,$$
$$\log 25299 = 4 \cdot 4030134, L \cos 26° 33' 54'' = 9 \cdot 9515452,$$
and $$L \tan 26° 33' 54'' = 9 \cdot 6989700.$$

17. Two sides of a triangle are 237 and 158 cm and the contained angle is 66° 40′; find the base and the other angles, having given

$$\log 2 = \cdot 30103, \log 79 = 1 \cdot 89763,$$
$$\log 22687 = 4 \cdot 35578, \quad L \cot 33° 20' = 10 \cdot 18197,$$
$$L \sin 33° 20' = 9 \cdot 73998, \quad L \tan 16° 54' = 9 \cdot 48262,$$
$$L \tan 16° 55' = 9 \cdot 48308, L \sec 16° 54' = 10 \cdot 01917,$$
and $$L \sec 16° 55' = 10 \cdot 01921$$

$$\left[\textit{Use the formula } \cos \frac{B - C}{2} = \frac{b + c}{2} \sin \frac{A}{2}.\right]$$

In the following four examples, the required logarithms must be taken from the tables.

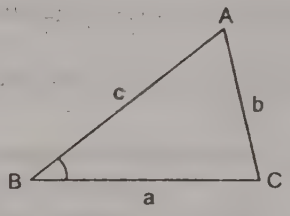
18. If $a = 242 \cdot 5, b = 164 \cdot 3$, and $c = 54° 36'$, solve the triangle.

19. If $b = 130, c = 63$, and $A = 42° 15' 30''$, solve the triangle.

20. Two sides of a triangle being 2265·4 and 1779 cm, and the included angle 58° 17′, find the remaining angles.

21. Two sides of a triangle being 237·09 and 130·96 cm, and the included angle 57° 59′, find the remaining angles.

184. Case III. *Given two sides b and c and the angle B opposite to one of them.*



The angle C is given by the relation

$$\frac{\sin C}{c} = \frac{\sin B}{b},$$

i.e.
$$\sin C = \frac{c}{b} \sin B \quad ...(1)$$

Taking logarithms, we determine C, and then A $(= 180° - B - C)$ is found.

The remaining side a is then found from the relation

$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

i.e.
$$a = b \frac{\sin A}{\sin B} \quad ...(2)$$

185. The equation (1) of the previous article gives in some cases no value, in some cases one, and sometimes two values, for C.

First, let B be an acute angle.

(α) If $b < c \sin B$, the right-hand member of (1) is greater than unity, and hence there is no corresponding value for C

(β) If $b = c \sin B$, the right-hand member of (1) is equal to unity and the corresponding value of C is 90°

(γ) If $b > c \sin B$, there are two values of C having $\frac{c \sin B}{b}$ as its sine, one value lying between 0° and 90° and the other between 90° and 180°.

Both of these values are not however always admissible.

For if $b > c$, then $B > C$. The obtuse-angled value of C is now not admissible; for, in this case, C cannot be obtuse unless B is obtuse also, and it is manifestly impossible to have *two* obtuse angles in a triangle.

If $b < c$ and B is an acute angle, both values of C are admissible. Hence there are two values found for A, and hence the relation (2) gives two values for a. In this case there are therefore two triangles satisfying the given conditions.

Secondly, let B be an obtuse angle.

If b is $<$ or $= c$, then B would be less than, or equal to, C, so that C would be an obtuse angle. The triangle would then be impossible.

If b is $> c$, the acute value of C, as determined from (1), would be admissible, but note the obtuse value. We should therefore only have one admissible solution.

Since, for some values of b, c and B, there is a doubt or ambiguity in the determination of the triangle, this case is called the **Ambiguous Case** of the solution of triangles.

186. *The Ambiguous Case may also be discussed in a geometrical manner.*

Suppose we were given the elements b, c and B, and that we proceeded to construct, or attempted to construct, the triangle.

We first measure an angle ABD equal to the given angle B.

We then measure along BA a distance BA equal to the given distance c, and thus determine the angular point A.

We have now to find a third point C, which must lie on BD and must also be such that its distance from A shall be equal to b.

To obtain it, we describe with centre A a circle whose radius is b.

The point or points, if any, in which this circle meets BD will determine the position of C.

Draw AD perpendicular to BD, so that

$$AD = AB \sin B = c \sin B.$$

One of the following events will happen.

The circle may not reach BD (Fig. 1) or it may touch BD (Fig. 2), or it may meet BD in two points C_1 and C_2 (Figs, 3 and 4).

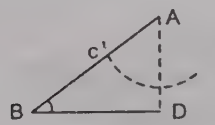


Fig. 1

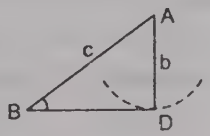


Fig. 2

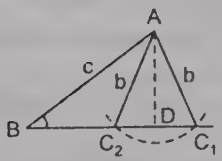


Fig. 3

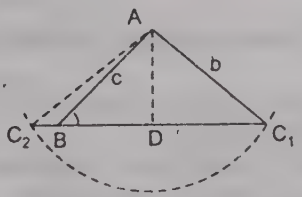


Fig. 4

In the case of Fig. 1, it is clear that there is no triangle satisfying the given condition.

Here $b < AD$, *i.e.* $< c \sin B$.

In the case of Fig. 2, there is one triangle ABD which is right-angled at D. Here

$$b = AD = c \sin B.$$

In the case of Fig. 3, there are two triangles ABC_1 and ABC_2. Here b lies in magnitude between AD and c, i.e. b is $> c \sin B$ and $< c$.

In the case of Fig. 4, there is only one triangle ABC_1 satisfying the given conditions [the triangle ABC_2 is inadmissible; for its angle at B is not equal to B but is equal to $180° - B$]. Here b is greater than both $c \sin B$ and c.

In the case when B is obtuse, the proper figures should be drawn. It will then be seen that when $b < c$ there is no triangle (for in the corresponding

triangles ABC_1 and ABC_2 the angle at B will be $180° - B$ and not B). If $b > c$, it will be seen that there is one triangle, and only one, satisfying the given conditions.

To sum up :

Given the elements b, c, and B of a triangle,

(α) If b is $< c \sin B$, there is no triangle.

(β) If $b = c \sin B$ and B is acute, there is one triangle right-angled.

(γ) If b is $> c \sin B$ and $< c$ and B is acute, there are two triangles satisfying the given conditions.

(δ) If b is $> c$, there is only one triangle.

Clearly if $b = c$, the points B and C_2 if Fig. 3. coincide and there is only one triangle.

(ε) If B is obtuse, there is no triangle except when $b > c$.

187. *The ambiguous case may also be considered algebraically as follows.*

From the figure of Art. 184, we have

$$b^2 = c^2 + a^2 - 2ca \cos B.$$

$$\therefore \quad a^2 - 2ac \cos B + c^2 \cos^2 B = b^2 - c^2 + c^2 \cos^2 B$$

$$= b^2 - c^2 \sin^2 B.$$

$$\therefore \quad a - c \cos B = \pm \sqrt{b^2 - c^2 \sin^2 B},$$

i.e.
$$a = c \cos B \pm \sqrt{b^2 - c^2 \sin^2 B} \qquad \ldots(1)$$

Now (1) is an equation to determine the value of a when b, c, and B are given.

(α) If $b < c \sin B$, the quantity $\sqrt{b^2 - c^2 \sin^2 B}$ is imaginary, and (1) gives no real value for a.

(β) If $b = c \sin B$, there is only one value, $c \cos B$, for a; there is thus only one triangle which is right-angled.

(γ) if $b > c \sin B$, there are two values for a. But, since a must be positive, the value obtained by taking the lower sign affixed to the radical is inadmissible unless

$$c \cos B - \sqrt{b^2 - c^2 \sin^2 B} \text{ is positive,}$$

i.e. unless
$$\sqrt{b^2 - c^2 \sin^2 B} < c \cos B,$$

i.e. unless
$$b^2 - c^2 \sin^2 B < c^2 \cos^2 B,$$

i.e. unless
$$b^2 < c^2.$$

There are therefore two triangles only when b is $> c \sin B$ and at the same time $< c$.

(δ) If B is an obtuse angle, then $c \cos B$ is negative, and one value of a is always negative and the corresponding triangle impossible.

The other value will be positive only when

$$c \cos B + \sqrt{b^2 - c^2 \sin^2 B} \text{ is positive,}$$

i.e. only when
$$\sqrt{b^2 - c^2 \sin^2 B} > -c \cos B.$$

i.e. only when $b^2 > c^2 \sin^2 B + c^2 \cos^2 B$,

i.e. only when $b > c$.

Hence, B being obtuse, there is no triangle if $b < c$, and only one triangle when $b > c$.

188. Ex. *Given* $b = 16, c = 25$, *and* $B = 33° 15'$, *prove that the triangle is ambiguous and find the other angles, using the five-figure tables at the end of this book.*

We have

$$\sin C = \frac{c}{b} \sin B \quad \frac{25}{16} \sin B = \frac{100}{64} \sin B \quad \frac{10^2}{2^6} \sin 33° 15'$$

Hence $L \sin C = 2 + L \sin 33° 15' - 6 \log 2$

$= 2 + 9{\cdot}73901 - 1{\cdot}80618$

$= 9{\cdot}93283.$

From the tables on the page xxxvi we see that this lies between $L \sin 58° 50'$

$L \sin C = 9{\cdot}93283$	$L \sin 59° = 9{\cdot}93307$
$L \sin 58° 50' = 9{\cdot}93230$	$L \sin 58° 50' = 9{\cdot}93230$
Diff. $= .00053$	Diff. for $10' = {\cdot}00077$

$\therefore$ angular diff. $= \frac{53}{77} \times 10'$

$= 6' 53''$ nearly.

$\therefore$ $C = 58° 56' 53''$, or $180° - 58° 56' 53''$.

Hence (Fig. 3, Art. 186) we have

$C_1 = 58° 56' 53''$ and $C_2 = 121° 3' 7''$.

$\therefore$ $\angle BAC_1 = 180° - 33° 15' - 58° 56' 53'' = 87° 48' 7''$,

and $\angle BAC_2 = 180° - 33° 15' - 121° 3' 7'' = 25° 41' 53''$

If we had used seven-figure tables we should have obtained the more accurate results $C_1 = 58° 56' 56''$ and $C_2 = 121° 3' 4''$.

EXAMPLE. XXXI.

[*The student should verify the results of some of the following examples (e.g. Nos. 3, 5, 6, 8, 9, 10, 12, 13) by an accurate graph.*]

1. If $a = 5, b = 7$, and $\sin A = \frac{3}{4}$, is there any ambiguity ?
2. If $a = 2, c = \sqrt{3} + 1$, and $A = 45°$, solve the triangle.
3. If $a = 100, c = 100\sqrt{2}$, and $A = 30°$, solve the triangle.
4. If $2b = 3a$, and $\tan^2 A = \frac{3}{5}$, prove that there are two values to the third side, one of which is double the other.
5. If $A = 30°, b = 8$, and $a = 6$, find c.

6. Given $B = 30°, c = 150$, and $b = 50\sqrt{3}$, prove that of the two triangles which satisfy the data one will be isosceles and the other right-angled. Find the greater value of the third side.
Would the solution have been ambiguous had
$B = 30°, c = 150$, and $b = 75$?

7. In the ambiguous case, given a, b, and A, prove that the difference between the two values of c is $2\sqrt{a^2 - b^2 \sin^2 A}$.

8. If $a = 5, b = 4$, and $A = 45°$, find the other angles, having given
$\log 2 = \cdot 30103$, $L \sin 34° 26' = 9 \cdot 7523919$,
and $L \sin 34° 27' = 9 \cdot 7525761$.

9. If $a = 9, b = 12$, and $A = 30°$, find c, having given
$\log 2 = \cdot 30103$, $\log 3 = \cdot 47712$,
$\log 171 = 2 \cdot 23301$, $\log 368 = 2 \cdot 56635$,
$L \sin 11° 48' 39'' = 9 \cdot 31108$, $L \sin 41° 48' 39'' = 9 \cdot 82391$,
and $L \sin 108° 11' 21'' = 9 \cdot 97774$.

10. Point out whether or no the solutions of the following triangles are ambiguous.
Find the smaller value of the third side in the ambiguous case and the other angles in both cases.
(1) $A = 30°, c = 250$ metres, and $a = 125$ metres;
(2) $A = 30°, c = 250$ metres, and $a = 200$ metres.
Given $\log 2 = \cdot 30103$, $\log 6 \cdot 03893 = \cdot 7809601$,
$L \sin 38° 41' = 9 \cdot 7958800$,
and $L \sin 8° 41' = 9 \cdot 1789001$.

11. Given $a = 250, b = 240$, and $A = 72° 4' 48''$, find the angles B and C, and state whether they can have more than one value, given
$\log 2 \cdot 5 = \cdot 3979400$, $\log 2 \cdot 4 = \cdot 3802112$,
$L \sin 72° 4' = 9 \cdot 9783702$, $L \sin 72° 5' = 9 \cdot 9784111$,
and $L \sin 65° 59' = 9 \cdot 9606739$.

12. Two straight roads intersect at an angle of 30°; from the point of junction two pedestrains A and B start at the same time, A walking along one road at the rate of 5 km per hour and B walking uniformly along the other road. At the end of 3 hours they are 9 km apart. Shew that there are two rates at which B may walk to fulfil this condition and find them.

In the following examples the required logarithms must be taken from the tables.

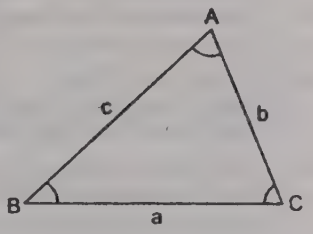
13. Two sides of a triangle are 1015 cm and 732 cm, and the angle opposite the latter side is 40°; find the angle opposite the former and prove that more than one value is admissible.

14. Two sides of a triangle being 5374·5 and 1586·6 metres, and the angle opposite the latter being 15° 11′, calculate the other angles of the triangle or triangles.

15. Given $A = 10°, a = 2308 \cdot 7$, and $b = 7903 \cdot 2$, find the smaller value of c.

189. *Case IV. Given one side and two angle, viz. a, B, and C*

Since the three angles of a triangle the together equal to two right angles, the third angle is given also.



The side b and c are now obtained from the relations

$$\frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a}{\sin A}$$

giving $$b = a\frac{\sin B}{\sin A},$$

and $$c = a\frac{\sin C}{\sin A}.$$

190. Case V. *The three angles A, B, and C given.*

Here the ratios only of the sides can be determined by the formulae

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Their absolute magnitudes cannot be found.

EXAMPLES. XXXII.

1. If $\cos A = \frac{17}{22}$ and $\cos C = \frac{1}{14}$, find the ratio of $a : b : c$.
2. The angles of a triangle are as $1 : 2 : 7$; prove that the ratio of the greatest side to the least side is $\sqrt{5} + 1 : \sqrt{5} - 1$.
3. If $A = 45°$, $B = 75°$, and $C = 60°$, prove that $a + c\sqrt{2} = 2b$.
4. Two angles of a triangle are 41° 13′ 22″ and 71° 19′ 5″ and the side opposite the first angle is 55; find the side opposite the latter angle, given $\log 55 = 1{\cdot}7403627$, $\log 79063 = 4{\cdot}8979775$,
   $L \sin 41° 13' 22'' = 9{\cdot}8188779$,
   and $L \sin 71° 19' 5'' = 9{\cdot}9764927$.
5. From each of two ships, one km apart, the angle is observed which is subtended by the other ship and a beacon on shore; these angle are found to be 52° 25′ 15″ and 75° 9′ 30″ respectively. Given
   $L \sin 75° 9' 30'' = 9{\cdot}9852635$,
   $L \sin 52° 25' 15'' = 9{\cdot}8990055$, $\log 1{\cdot}2197 = {\cdot}0862530$,
   and $\log 1{\cdot}2198 = 0862886$,
   find the distance of the beacon from each of the ships.
6. The base angles of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$; prove that the base is equal to twice the height.

For the following the examples the required logarithms must be taken from the tables.

7. The base of a triangle being 7 cm and the base angles 129° 23′ and 38° 36′, find the length of its shorter side.

8. If the angles of a triangle be as 5 : 10 : 21, and the side opposite the smaller angle be 3 cm, find the other sides.
9. The angles of a triangle being 150° 18° 20′, and 11° 40′ and the longest side being 10 metres, find the length of the shortest side.
10. To get the distance of a point A from a point B, a line BC and the angles ABC and BCA are measured, and are found to be 287 metres and 55° 32′ 10″ and 51° 8′ 20″ respectively. Find the distance AB.
11. To find the distance from A to P a distance. AB, of 1 km is measured in a convenient direction. At A the angle PAB is found to be 41° 18′ and at B the angle PBA is found to be 114° 38′. What is the required distance to the nearest metre ?

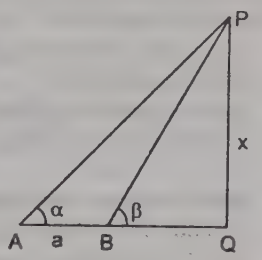
CHAPTER XIV

HEIGHTS AND DISTANCES

191. In the present chapter we shall consider some questions of the kind which occur in land-surveying. Simple questions of this kind have already been considered in Chapter III.

192. *To find the height of an inaccessible tower by means of observations made at distant points.*

Suppose PQ to be the tower and that the ground passing through the foot Q of the tower is horizontal. At a point A on this ground measure the angle of elevation α of the top of the tower.



Measure off a distance $AB\ (=a)$ from A directly toward the foot of the tower, and at B measure the angle of elevation β.

To find the unknown height x of the tower, we have to connect it with the measured length a. This is best done as follows:

From the triangle PBQ, we have

$$\frac{x}{BP} = \sin\beta \qquad \ldots(1),$$

and, from the triangle PAB, we have

$$\frac{PB}{a} = \frac{\sin PAB}{\sin BPA} = \frac{\sin\alpha}{\sin(\beta-\alpha)} \qquad \ldots(2),$$

since $\angle BPA = \angle QBP - \angle QAP = \beta - \alpha$.

From (1) and (2), by multiplication, we have

$$\frac{x}{a} = \frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)},$$

i.e.

$$x = a\,\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$$

The height x is therefore given in a form suitable for logarithmic calculation.

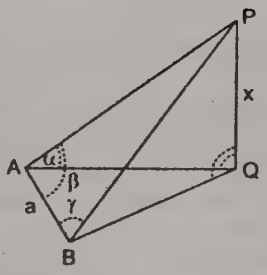
Numerical Example. If $a = 100$ *metres,* $\alpha = 30°$, *and* $\beta = 60°$, *then*

$$x = 100\,\frac{\sin 30°\sin 60°}{\sin 30°} = 100 \times \frac{\sqrt{3}}{2} = 86.6 \text{ metres.}$$

193. It is often not convenient to measure AB directly towards Q.

Measure therefore AB in any other suitable direction on the horizontal ground, and at A measure the angle of elevation α of P, and also the angle PAB $(=\beta)$.

At B measure the angle PBA $(=\gamma)$.



In the triangle PAB, we have then

$$\angle APB = 180° - \angle PAB - \angle PBA = 180° - (\beta + \gamma).$$

Hence $$\frac{AP}{a} = \frac{\sin PBA}{\sin BPA} = \frac{\sin \gamma}{\sin (\beta + \gamma)}.$$

From the triangle PAQ, we have

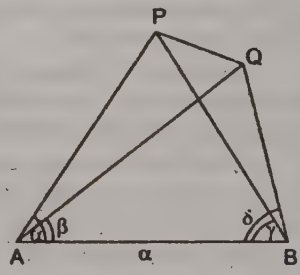
$$x = AP \sin \alpha = a \frac{\sin \alpha \sin \gamma}{\sin (\beta + \gamma)}.$$

Hence x is found by an expression suitable for logarithmic calculation.

194. *To find the distance between two inaccessible points by means of observations made at two points the distance between which is known, all four points being supposed to be in one plane.*

Let P and Q be two points whose distance apart, PQ, is required.

Let A and B be the two known points whose distance apart, AB, is given to be equal to a.



At A measure the angles PAB and QAB, and let them be α and β respectively.

At B measure the angle PBA and QBA, and let them by γ and δ respectively.

Then in the triangle PAB we have one side a and the two adjacent angles α and γ given, so that, as in Art. 163, we have AP given by the relation

$$\frac{AP}{a} = \frac{\sin \gamma}{\sin APB} = \frac{\sin \gamma}{\sin (\alpha + \gamma)} \qquad \ldots(1)$$

In the triangle QAB we have, similarly,

$$\frac{AQ}{a} = \frac{\sin \delta}{\sin (\beta + \delta)} \qquad \ldots(2)$$

In the triangle APQ we have now determined the sides AP and AQ; also the included angle PAQ $(= \alpha - \beta)$ is known. We can therefore find the side PQ by the method of Art. 181.

If the four points $A, B, P,$ and Q be not in the same plane, we must, in addition, measure the angle PAQ; for in this case PAQ is not equal to $\alpha - \beta$. In other respects the solution will be the same as above.

195. Bearings and Points of the Compass. The bearing of a given point B as seen from a given point O is the direction in which B is seen from O. Thus if the direction of OB bisect the angle between East and North, the bearing of B is said to be North-East.

If a line is said to bear 20° West of North, we mean that it is inclined to the North direction at an angle of 20°, this angle being measured from the North towards the West.

To facilitate the statement of the bearing of a point the circumference of the mariner's compass-card is divided into 32 equal portions, as in the above figure, and the subdivisions marked as indicated. Consider only the quadrant between East and North. The middle point of the arc between N. and E. is marked North-East (N.E.). The bisectors of the arcs between N.E. and N. and E. are respectively called North-North-East and East-North-East (N.N.E. and E.N.E.). The other four subdivisions, reekoning from N., are called North by East, N.E. by North, N.E. by East, and East by North. Similarly the other three quadrants are subdivided.

It is clear that the arc between two subdivisions of the card subtends an angle of $\frac{360°}{32}$, *i.e.*, $11\frac{1}{4}°$, at the centre O.

EXAMPLES. XXXIII.

1. A flagstaff stands on the middle of a square tower. A man on the ground, opposite the middle of one face and distant from it 100 metres, just sees the flag; on his receding another 100 feet, the tangents of elevation of the top of the tower and the top of the flagstaff are found to be $\frac{1}{2}$ and $\frac{5}{9}$. Find the dimensions of the tower and the height of the flagstaff, the ground being horizontal.

2. A man, walking on a level plane towards a tower, observes that at a certain point the angular height of the tower is 10°, and, after going 50 metres nearer the tower, the elevation is found to be 15°. Having given
$L \sin 15° = 9.4129962$, $L \cos 5° = 9.9983442$,
$\log 25.783 = 1.4113334$, and $\log 25.784 = 1.4113503$,
find to 4 places of decimals the height of the tower in metres.

3. DE is a tower standing on a horizontal plane and ABCD is a straight line in the plane. The height of the tower subtends an angle θ at 2θ at B, and 3θ at C. If AB and BC be respectively 50 and 20 metres, find the height of the tower and the distance CD.
4. A tower, 50 metres high, stands on the top of a mound; from a point on the ground the angles of elevation of the top and bottom of the tower are found to be 75° and 45° respectively; find the height of the mound.
5. A vertical pole (more than 10 metres high) consists of two parts, the lower being $\frac{1}{3}rd$ of the whole. From a point in a horizontal plane through the foot of the pole and 40 metres from it, the upper part subtends an angle whose tangent is $\frac{1}{2}$. Find the height of the pole.
6. A tower subtends an angle a at a point on the same level as the foot of the tower, and at a second point, h metres above the first, the depression of the foot of the tower is β. Find the height of the tower.
7. A person in a balloon, which has ascended vertically from flat land at the sea-level, observes the angle of depression of a ship at anchor to be 30°; after descending vertically for 600 metres, he finds the angle of depression to be 15°; find the horizontal distance of the ship from the point of ascent.
8. PQ is a tower standing on a horizontal plane, Q being its foot; A and B are two points on the plane such that the $\angle QAB$ is 90°, and AB is 40 metres. It is found that

   $$\cot PAQ = \frac{3}{10} \text{ and } \cot PBQ = \frac{1}{2}.$$

   Find the height of the tower.
9. A coloumn is E.S.E. of an observer, and at noon the end of the shadow is North-East of him. The shadow is 80 metres long and the elevation of the column at the observer's station is 45°. Find the height of the column.
10. A tower is observed from two stations A and B. It is found to be due north of A and north-west of B. B is due east of A and distant from it 10 metres. The elevation of the tower as seen from A is the complement of the elevation as seen from B. Find the height of the tower.
11. The elevation of a steeple at a place due south of it is 45° and at another place due west of the former place the elevation is 15°. If the distance between the two places be a, prove that the height of the steeple is

    $$\frac{a(\sqrt{3}-1)}{2\sqrt{3}}.$$

12. A person stands in the diagonal produced of the square base of a church tower, at a distance $2a$ from it, and observes the angles of elevation of each of the two outer corners of the top of the tower to be 30°, whilst that of the nearest corner is 45°. Prove that the breadth of the tower is $a(\sqrt{10}-\sqrt{2})$.

13. A person standing at a point A due south of a tower built on a horizontal plane observes the altitude of the tower to be 60°. He then walks to B due west of A and observes the altitude to be 45°, and again at C in AB produced he observes it to be 30°. Prove that B is midway between A and C.

14. At each end of a horizontal base of length $2a$ it is found that the angular height of a certain peak is θ and that at the middle point it is ϕ. Prove that the vertical height of the peak is

$$\frac{a \sin\theta \sin\phi}{\sqrt{\sin(\phi+\theta)\sin(\phi-\theta)}}.$$

15. A and B are two stations 1000 m apart; P and Q are two stations in the same plane as AB and on the same side of it; the angles PAB, PBA, QAB, and QBA are respectively 75°, 30°, 45° and 90°; find how far P is from Q and how far each is from A and B.

For the following seven examples the required logarithms must be taken from the tables.

16. At a point on a horizontal plane the elevation of the summit of a mountain is found to be 22° 15′, and at another point on the plane, 1 km farther away in a direct line, its elevation is 10° 12′; find the height of the mountain.

17. From the top of a hill the angles of depression of two successive points, 1 km apart on level ground and in the same vertical plane with the observe, are found to be 5° and 10° respectively. Find the height of the hill and the horizontal distance to the nearest point.

18. A cliff and a tower stand on the same horizontal plane. The height of the tower is 140 metres, and the angles of depression of the top and bottom of the tower as seen from the top of the cliff are 40° and 80° respectively. Find the height of the tower.

19. A tower PN stands on level ground. A base AB is measured at right angles to AN, the points A, B, and N being in the same horizontal plane, and the angles PAN and PBN are found to be α and β respectively. Prove that the height of the tower is

$$AB\frac{\sin\alpha\sin\beta}{\sqrt{\sin(\alpha-\beta)\sin(\alpha+\beta)}}.$$

If $AB = 100$ m, $\alpha = 70°$, and $\beta = 50°$, calculate the height.

20. A man, standing due south of a tower on a horizontal plane through its foot, finds the elevation of the top of the tower to be 54° 16′; he goes east 100 metres and finds the elevation to be then 50° 8′. Find the height of the tower.

21. A man in a balloon observes that the angle of depression of an object on the ground bearing due north is 33°; the balloon drifts 3 miles due west and the angle of depression is now found to be 21°. Find the height of the balloon.

22. From the extremities of a horizontal base-line AB, whose length is 1 km, the bearings of the foot C of a tower are observed and it is found that $\angle CAB = 56° 23'$, $\angle CBA = 47° 15'$, and the elevation of the tower from A is 9° 25′; find the height of the tower.

196. Ex. 1. *A flagstaff is on the top of a tower which stands on a horizontal plane. A person observes the angles, a and β, subtended at a point on the horizontal plane by the flagstaff and the tower; he then walks a known distance a toward the tower and finds that the flagstaff subtends the same angle as before; prove that the height of the tower and the length of the flagstaff are respectively*

$$\frac{a \sin\beta \cos(\alpha+\beta)}{\cos(\alpha+2\beta)} \text{ and } \frac{a \sin\alpha}{\cos(\alpha+2\beta)}.$$

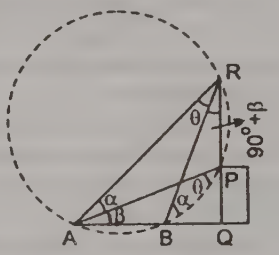
Let P and Q be the top and foot of the tower, and let PR be the flagstaff. Let A and B be the points at which the measurements are taken, so that $\angle PAQ = \beta$ and $\angle PAR = \angle PBR = \alpha$. Since the two latter angles are equal, a circle will go through the four points A, B, P, and R.

To find the height of the flagstaff we have to connect the unknown length PR with the known length AB.

This may be done by connecting each with the length AR.

To do this, we must first determine the angles of the triangles ARP and ARB.

Since A, B, P, and R lie on a circle, we have



$$\angle BRP = \angle BAP = \beta,$$

and $\angle APB = \angle ARB = \theta$ (say).

Also $\angle APR = 90° + \angle PAQ = 90° + \beta$.

Hence, since the angles of the triangle APR are together equal to two right angles, we have

$$180° = \alpha + (90° + \beta) + (\theta + \beta),$$

so that $$\theta = 90° - (\alpha + 2\beta) \qquad \ldots(1)$$

From the triangles APR and ABR we then have

$$\frac{PR}{\sin\alpha} = \frac{AR}{\sin RPA} = \frac{AR}{\sin RBA} = \frac{a}{\sin\theta} \qquad \text{(Art. 163).}$$

[It will be found in Chap. XV that each of these quantities is equal to the diameter of the circle.]

Hence the height of the flagstaff

$$= PR = \frac{a \sin\alpha}{\sin\theta} = \frac{a \sin\alpha}{\cos(\alpha+2\beta)}, \text{ by (1).}$$

Again, $$\frac{PQ}{PB} = \cos BPQ = \cos(\alpha+\beta) \qquad \ldots(2)$$

and $$\frac{PB}{a} = \frac{\sin PAB}{\sin APB} = \frac{\sin\beta}{\sin\theta} \qquad \ldots(3)$$

Hence, from (2) and (3), by multiplication,

$$\frac{PQ}{a} = \frac{\sin\beta \cos(\alpha+\beta)}{\sin\theta} = \frac{\sin\beta \cos(\alpha+\beta)}{\cos(\alpha+2\beta)}, \text{ by (1).}$$

Also, $BQ = PQ \tan BPQ = PQ \tan(\alpha+\beta)$

$$= a\frac{\sin\beta\sin(\alpha+\beta)}{\cos(\alpha+2\beta)},$$

and $$AQ = a + BQ = a\frac{\cos(\alpha+2\beta)+\sin\beta\sin(\alpha+\beta)}{\cos(\alpha+2\beta)}$$

$$= a\frac{\cos\beta\cos(\alpha+\beta)}{\cos(\alpha+2\beta)}.$$

If a, α and β be given numerically, these results are all in a form suitable for logarithmic computation.

Ex. 2. *At a distance a from the foot of a tower AB, of known height b, a flagstaff BC and the tower subtend equal angles. Find the height of the flagstaff.*

Let O be the point of observation, and let the angles AOB and BOC be each θ; also let the height BC be y.

We then have $\tan\theta = \frac{b}{a}$, and $\tan 2\theta = \frac{b+y}{a}$.

Hence $$\frac{b+y}{a} = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\frac{b}{a}}{1-\frac{b^2}{a^2}},$$

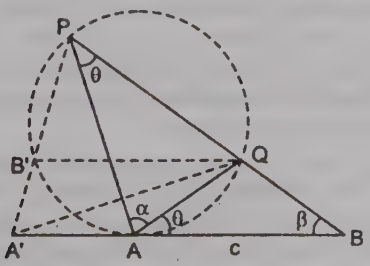
so that $$\frac{b+y}{a} = \frac{2ab}{a^2-b^2}.$$

Then $$y = \frac{2a^2b}{a^2-b^2} - b = b\frac{a^2+b^2}{a^2-b^2}.$$

If a and b be given numerically, we thus easily obtain y.

197. Ex. *A man walks along a straight road and observes that the greatest angle subtended by two objects is α; from the point where this greatest angle is subtended he walks a distance c along the road, and finds that the two objects are now in a straight line which makes an angle β with the road; prove that the distance between the object is*

$$c\sin\alpha\sin\beta\sec\frac{\alpha+\beta}{2}\sec\frac{\alpha-\beta}{2}.$$



Let P and Q be the two points, and let PQ meet the road in B.

If A be the point at which the greatest angle is subtended, then A must be the point where a circle drawn through P and Q touches the road.

[For, take any other point A' on AB, and join it to P cutting the circle in B', and join $A'Q$ and $B'Q$.

Then $\angle PA'Q < \angle PB'Q$

and therefore $< \angle PAQ$.]

Let the angle QAB be called θ. Then the angle APQ is θ also.

Hence $180° =$ sum of the angles of the triangle PAB

$$= \theta + (\alpha + \theta) + \beta,$$

so that $\theta = 90° - \dfrac{\alpha + \beta}{2}$.

From the triangles PAQ and QAB we have

$$\frac{PQ}{AQ} = \frac{\sin\alpha}{\sin\theta}, \text{ and } \frac{AQ}{c} = \frac{\sin\beta}{\sin AQB} = \frac{\sin\beta}{\sin(\theta + \alpha)}.$$

Hence, by multiplication, we have

$$\frac{PQ}{c} = \frac{\sin\alpha\sin\beta}{\sin\theta\sin(\theta + \alpha)}$$

$$= \frac{\sin\alpha\sin\beta}{\cos\dfrac{\alpha + \beta}{2}\cos\dfrac{\alpha - \beta}{2}}.$$

$$\therefore \quad PQ = c\sin\alpha\sin\beta\sec\frac{\alpha + \beta}{2}\sec\frac{\alpha - \beta}{2}.$$

EXAMPLES. XXXIV.

1. A bridge has 5 equal spans, each of 10 m measured from the centre of the piers, and a boat is moored in a line with one of the middle piers. The whole length of the bridge subtends a right angle as seen from the boat. Prove that the distance of the boat from the bridge is $10\sqrt{6}$ m.

2. A ladder placed at an angle of 75° with the ground just reaches the sill of a window at a height of 9 metres above the ground on one side of a street. On turning the ladder over without moving its foot, it is found that when it rests against a wall on the other side of the street it is at an angle of 15° with the ground. Prove that the breadth of the street and the length of the ladder are respectively $9(3 - \sqrt{3})$ and $9(\sqrt{6} - \sqrt{2})$ metres.

3. From a house on one side of a street observations are made of the angle subtended by the height of the opposite house; from the level of the street, the angle subtended is the angle whose tangent is 3; from two windows one above the other the angle subtended is found to be the angle whose tangent is -3; the height of the opposite house being 30 metres, find the height above the street of each of the two windows.

4. A rod of given length can turn in a vertical plane passing through the sun, one end being fixed on the ground; find the longest shadow it can cast on the ground.
   Calculate the altitude of the sun when the longest shadow it can cast is $3\frac{1}{2}$ times the length of the rod.
5. A person on a ship A observes another ship B leaving a harbour, whose bearing is then N.W. After 10 minutes A, having sailed one km. N.E., sees B due west and the harbour then bears 60° west of north. After another 10 minutes B is observed to bear S.W. Find the distances between A and B at the first observation and also the direction and rate of B.
6. A person on a ship sailing north sees two lighthouses, which are 6 km. apart, in a line due west; after an hour's sailing one of them bears S.W. and the other S.S.W. Find the ship's rate.
7. A person on a ship sees a lighthouse N.W. of himself. After sailing for 12 km. in a direction 15° south of W. the lighthouse is seen due N. Find the distance of the lighthouse from the ship in each position.
8. A man, travelling west along a straight road, observes that when he is due south of a certain windmill the straight line drawn to a distant tower makes an angle of 30° with the road. One km. farther on the bearings of the windmill and tower are respectively N.E. and N.W. Find the distances of the tower from the windmill and from the nearest point of the road.
9. An observer on a headland sees a ship due north of him; after a quarter of an hour he sees it due east and after another half-hour he sees it due south-east; find the angle that the ship's course makes with the meridian and the time after the ship is first seen until it is nearest the observer, supposing that it sails uniformly in a straight line.
10. A man walking along a straight road, which runs in a direction 30° east of north, notes when he is due south of a certain house; when he has walked 1 km. farther, he observes that the house lies due west and that a windmill on the opposite side of the road is N.E. of him; three km farther on he finds that he is due north of the windmill; prove that the line joining the house and the windmill makes with the road the angle whose tangent is
$$\frac{48 - 25\sqrt{3}}{11}.$$
11. A, B and C are three consecutive milestones on a straight road from each of which a distant spire is visible. The spire is observed to bear north-east at A, east at B, and 60° east of south at C. Prove that the shortest distance of the spire from the road is $\frac{7 + 5\sqrt{3}}{13}$ km.
12. Two stations due south of a tower, which leans towards the north, are at distances α and b from its foot; it α and β are the elevations of the top of the tower from these stations, prove that its inclination to the horizontal is

$$\cot^{-1}\frac{b\cot\alpha - a\cot\beta}{b-a}.$$

13. From a point A on a level plane the angle of elevation of a balloon is α, the balloon being south of A; from a point B, which is at a distance c south of A, the balloon is seen northwards at an elevation of β; find the distance of the balloon from A and its height above the ground.

14. A statue on the top of a pillar subtends the same angle α at distances of 9 and 11 m from the pillar; if $\tan\alpha = \frac{1}{10}$, find the height of the pillar and of the statue.

15. A flagstaff on the top of a tower is observed to subtend the same angle α at two points on a horizontal plane, which lie on a line passing through the centre of the base of the tower and whose distance from one another is $2a$, and an angle β at a point halfway between them. Prove that the height of the flagstaff is

$$a\sin\alpha\sqrt{\frac{2\sin\beta}{\cos\alpha\sin(\beta-\alpha)}}.$$

16. An observer in the first place stations himself at a distance a metres from a column standing upon a mound. He finds that the column subtends an angle, whose tangent is $\frac{1}{2}$, at his eye which may be supposed to be on the horizontal plane through the base of the mound. On moving $\frac{2}{3}a$ m nearer the column, he finds that the angle subtended is unchanged. Find the height of the mound and of the column.

17. A church tower stands on the bank of a river, which is 50 m. wide, and on the top of the tower is a spire 10 m. high. To an observer on the opposite bank of the river, the spire subtends the same angle that a pole 2 m. high subtends when placed upright on the ground at the foot of the tower. Prove that the height of the tower is nearly 95 m.

18. A person, wishing to ascertain the height of a tower, stations himself on a horizontal plane through its foot at a point at which the elevation of the top in 30°. On walking a distance a in a certain direction he finds that the elevation of the top is the same as before, and on then walking a distance $\frac{5}{3}a$ at right angles to his former direction he finds the elevation of the top to be 60°. Prove that the height of the tower is either $\sqrt{\frac{5}{6}a}$ or $\sqrt{\frac{85}{48}a}$.

19. The angles of elevation of the top of a tower, standing on a horizontal plane, from two points distant a m. and b m. from the base and in the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ m., and, if θ be the angle subtended at the top of the tower by the line joining the two points, then $\sin\theta = \frac{a\sim b}{a+b}$.

20. A tower 150 m. high stands on the top of a cliff 80 m. high. To an observer on a ship, the tower and cliff subtend equal angles. How far from the cliff is the observer if he is 5m. above the plane passing through the foot of the cliff ?

21. A statue on the top of a pillar, standing on level ground, is found to subtend the greatest angle a at the eye of an observer when his distance from the pillar is c m.; prove that the height of the statue is $2c \tan a$ m., and find the height of the pillar.

22. A tower stood at the foot of an inclined plane whose inclination to the horizon was 9°. A line 100 m. in length was measured straight up the incline from the foot of the tower, and at the end of this line the tower subtended an angle of 54°. Find the height of the tower, having given

$$\log 2 = \cdot 30103, \quad \log 114 \cdot 4123 = 2 \cdot 0584726,$$

and $L \sin 54° = 9 \cdot 9079576$.

23. A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 metres, and then finds that the tower subtends an angle of 30°. Prove that the height of the tower is $40(\sqrt{6} - \sqrt{2})$ metres.

24. The altitude of a certain rock is 47°, and after walking towards it 1 km up a slope inclined at 30° to the horizon an observer finds its altitude to be 77°. Find the vertical height of the rock above the first point of observation, given that $\sin 47° = \cdot 73135$.

25. A man observes that, when he has walked c metres up an inclined plane, the angular depression of an object in a horizontal plane through the foot of the slope is α, and that, when he has walked a further distance of c feet, the depression is β. Prove that the inclination of the slope to the horizon is the angle whose cotangent is $(2 \cot \beta - \cot \alpha)$.

26. A regular pyramid on a square base has an edge 150 m long, and the length of the side of its base is 200 m. Find the inclination of its face to the base.

27. A pyramid has for base a square of side a; its vertex lies on a line through the middle point of the base and perpendicular to it, and at a distance h from it; prove that the angle α between the two lateral faces is given by the equation

$$\sin \alpha = \frac{2h\sqrt{2a^2 + 4h^2}}{a^2 + 4h^2}.$$

28. A flagstaff, 10 metres high, stands in the centre of an equilateral triangle which is horizontal. From the top of the flagstaff each side subtends an angle of 60°; prove that the length of the side of the triangle is $5\sqrt{6}$ metres.

29. The extremity of the shadow of a flagstaff, which is 6 m high and stands on the top of a pyramid on a square base, just reaches the side of the base and is distant 56 and 8 m respectively from the extremities of that side. Find the sun's altitude if the height of the pyramid be 34 m.

30. The extremity of the shadow of a flagstaff, which is 6 metres high and stands on the top of a pyramid on a square base, just reaches the side of the base and is distant x feet and y feet respectively from the ends of that side; prove that the height of the pyramid is

$$\sqrt{\frac{x^2+y^2}{2}}\tan\alpha - 6,$$

where α is the elevation of the sun.

31. The angle of elevation of a cloud from a point h metres above a lake is α, and the angle of depression of its reflexion in the lake is β; prove that its height is $h\dfrac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)}$.

32. The shadow of a tower is observed to be half the known height of the tower and sometime afterwards it is equal to the known height; how much will the sun have gone down in the interval, given
$\log 2 = \cdot 30103$, $L\tan 63°\,26' = 10\cdot 3009994$,
and diff. for $1' = 3159$?

33. An isosceles triangle of wood is placed in a vertical plane, vertex upwards, and faces the sun. If $2a$ be the base of the triangle, h its height, and 30° the altitude of the sun, prove that the tangent of the angle at the apex of the shadow is $\dfrac{2ah\sqrt{3}}{3h^2-a^2}$.

34. A rectangular target faces due south, being vertical and standing on a horizontal plane. Compare the area of the target with that of its shadow on the ground when the sun is $\beta°$ from the south at an altitude of $\alpha°$.

35. A spherical ball, of diameter δ, subtends an angle α at a man's eye when the elevation of its centre is β; prove that the height of the centre of the ball is $\frac{1}{2}\delta\sin\beta\operatorname{cosec}\frac{\alpha}{2}$.

36. A man standing on a plane observes a row of equal and equidistant pillars, the 10th and 17th of which subtend the same angle that they would do if they were in the position of the first and were respectively $\frac{1}{2}$ and $\frac{1}{3}$ of their height. Prove that, neglecting the height of the man's eye, the line of pillars is inclined to the line drawn from his eye to the first at an angle whose secant is nearly 2·6.

For the following nine examples the required logarithms must be taken from the tables.

37. A and B are two points, which are on the banks of a river and opposite to one another, and between them is the mast, PN, of a ship; the breadth of the river is 100 metres, and the angular elevation of P at A is 14° 20′ and at B 8° 10′. What is the height of P above AB ?

38. AB is a line 1000 metres long; B is due north of A and from B a distant point P bears 70°east of north; at A it bears 41° 22′ east of north; find the distance from A to P.

39. A is a station exactly 10 km west of B. The bearing of a particular rock from A is 74° 19′ east of north, and its bearing from B is 26° 51′ west of north. How far is it north of the line AB ?

40. The summit of a spire is vertically over the middle point of a horizontal square enclosure whose side is of length a metres; the height of the spire is h metres above the level of the square. If the shadow of the spire just reaches a corner of the square when the sun has an altitude θ, prove that $h\sqrt{2} = a \tan\theta$.

Calculate h, having given $a = 100$ metres and $\theta = 25° 15'$.

41. Walking along a straight level road in a direction N.W., I notice two spires, P and Q, in a straight line with me on a bearing N. 20° E., P being the nearer spire. After walking 4 km. farther along the road, P bears E. 22° S. and Q bears E. 26° N. Find the distance between the spires.

42. AB is a road running uphill in a direction due east at an inclination of 10°, B being above A and at a distance of 500 metres from A. The bearing of an object C from A is E. 50° N. at an elevation of 43°, and the bearing of C from B is W. 65°N. Calculate the horizontal and vertical distances of C from A.

43. A wireless signal from an aeroplane is intercepted at two direction finding stations, A and B, which are five km. apart in a north and south line. From A the direction of the aeroplane is found to be 66° west of north, and from B it is found to be 20° west of south. At the same time the altitude of the aeroplane is observed from A to be 40°. Find the height of the aeroplane above A.

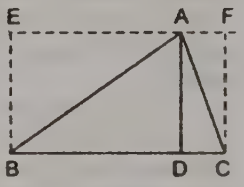
44. An observer sees an aeroplane due N. at an elevation of 8°. Two minutes later he sees it N.E. at the same elevation. It is known to be going due E., the horizontal component of its velocity being 80 km an hour. Shew that it is rising at the rate of nearly 77·615 metres per minute.

45. A small balloon is released from a point on a level plain, and ascends with a constant vertical velocity of 100 metres per minute. After it has risen for 10 minutes it is at P, and one minute later is at Q. At a station O on the plain the bearing of P is due east, and OP makes an angle of 63° with the plain. At O the bearing of Q is 29° north of east, and OQ makes an angle of 70° with the plain. Assuming the wind to blow with constant horizontal velocity, find this velocity, and show that its direction is about 50° 37′ north of west.

CHAPTER XV

PROPERTIES OF A TRIANGLE

198. Area of a given triangle. Let ABC be any triangle, and AD the perpendicular drawn from A upon the opposite side.



Through A draw EAF parallel to BC, and draw BE and CF perpendicular to it. By Geometry, the area of the triangle ABC

$$= \frac{1}{2} \text{ rectangle } BF = \frac{1}{2} BC \,.\, CF = \frac{1}{2} a \,.\, AD.$$

But $AD = AB \sin B = c \sin B$.

The area of the triangle ABC therefore $= \frac{1}{2} ca \sin B$.

This area is denoted by Δ.

Hence $\Delta = \frac{1}{2}$ **ca sin B** $= \frac{1}{2}$**ab sin C** $= \frac{1}{2}$**bc sin A** ...(1)

By Art. 169, we have $\sin A = \dfrac{2}{bc} \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$,

so that $\Delta = \frac{1}{2} bc \sin A = \sqrt{\mathbf{s\,(s-a)\,(s-b)\,(s-c)}}$ ...(2)

This latter quantity is often called S.

EXAMPLES. XXXV.

Find the area of the triangle ABC when

1. $a = 13$, $b = 14$, and $c = 15$.
2. $a = 18$, $b = 24$, and $c = 30$.
3. $a = 25$, $b = 52$, and $c = 63$.
4. $a = 125$, $b = 123$, and $c = 62$.
5. $a = 15$, $b = 36$, and $c = 39$.
6. $a = 287$, $b = 816$, and $c = 865$.
7. $a = 35$, $b = 84$, and $c = 91$.
8. $a = \sqrt{3}$, $b = \sqrt{2}$, and $c = \dfrac{\sqrt{6} + \sqrt{2}}{2}$
9. If $B = 45°$, $C = 60°$, and $a = 2(\sqrt{3} + 1)$ cm, prove that the area of the triangle is $6 + 2\sqrt{3}$ sq. cm.

10. The sides of a triangle are 119, 111, and 92 metres; prove that its area is 10 sq. yards less than an acre.
11. The sides of a triangular field are 242, 1212, and 1450 metres; prove that the area of the field is 29040 sq. m.
12. A workman is told to make a triangular enclosure of sides 50, 41, and 21 metres respectively; having made the first side one metre too long, what length must he make the other two sides in order to enclose the prescribed area with the prescribed length of fencing ?
13. Find, correct to ·0001 of a cm, the length of one of the equal sides of an isosceles triangle on a base of 14 cm having the same area as a triangle whose sides are 13·6, 15, and 15·4 cm.
14. If one angle of a triangle be 60°, the area $10\sqrt{3}$ square feet, and the perimeter 20 cm, find the lengths of the sides.
15. The sides of a triangle are in A.P. and its area is $\frac{3}{5}$ths of an equilateral triangle of the same perimeter; prove that its sides are in the ratio 3 : 5 : 7, and find the greatest angle of the triangle.
16. In a triangle the least angle is 45° and the tangents of the angles are in A.P. If its area be 27 square cm; prove that the lengths of the sides are $3\sqrt{5}$, $6\sqrt{2}$, and 9 cm, and that the tangents of the other angles are respectively 2 and 3.
17. The lengths of two sides of a triangle are 12 cm and $12\sqrt{2}$ cm respectively, and the angle opposite the shorter side in 30°; prove that there are two triangles satisfying these condition, find their angles, and shew that their areas are in the ratio
$$\sqrt{3}+1 : \sqrt{3}-1.$$
18. Find by the aid of the tables the area of the larger of the two triangles given by the data
$A = 31° 15'$, $a = 5$ cm., and $b = 7$ cm.

199. On the circles connected with a given triangle.

The circle which passes through the angular points of a triangle ABC is called its circumscribing circle or, more briefly, its **circumcircle**. The centre of this circle is found by the construction of Art. 200. Its radius is always called R.

The circle which can be inscribed within the triangle so as to touch each of the sides is called its inscribed circle or, more briefly, its **incircle**. The centre of this circle is found by the construction of Art. 202. Its radius will be denoted by r.

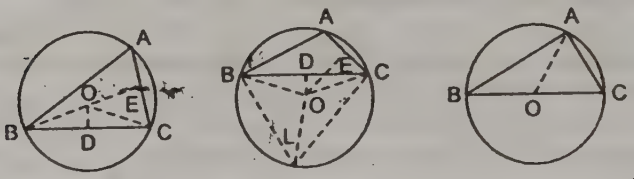
The circle which touches the side BC and the two sides AB and AC produced is called the escribed circle opposite the angle A. Its radius will be denoted by r_1.

Similarly r_2 denotes the radius of the circle which touches the side CA and the two sides BC and BA produced. Also r_3 denotes the radius of the circle touching AB and the two sides CA and CB produced.

200. *To find the magnitude of R, the radius of the circumcircle of any triangle ABC.*

Bisect the two sides BC and CA in D and E respectively, and draw DO and EO perpendicular to BC and CA.

By Geometry, O is the centre of the circumcircle. Join OB and OC.



The point O may either lie within the triangle as in Fig. 1, or without it as in Fig. 2, or upon one of the sides as in Fig. 3.

Taking the first figure, the two triangles BOD and COD are equal in all respects, so that

$$\angle BOD = \angle COD,$$

$$\therefore \quad \angle BOD = \tfrac{1}{2}\angle BOC = \angle BAC$$

$$= A.$$

Also $$BD = BO \sin BOD.$$

$$\therefore \quad \frac{a}{2} = R \sin A.$$

If A be obtuse, as in Fig. 2, we have

$$\angle BOD = \tfrac{1}{2}\angle BOC = \angle BLC = 180° - A,$$

so that, as before, $\sin BOD = \sin A$,

and $$R = \frac{a}{2 \sin A}.$$

If A be a right angle, as in Fig. 3, we have

$$R = OA = OC = \frac{a}{2}$$

$$= \frac{a}{2 \sin A}, \text{ since in this case } \sin A = 1.$$

The relation found above is therefore true for all triangles.

Hence, in all three cases, we have

$$\mathbf{R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}} \qquad \ldots(1)$$

201. *In Art. 169 we have shewn that*

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2S}{bc},$$

where S is the area of the triangle.

Substituting this value of $\sin A$ in (1), we have

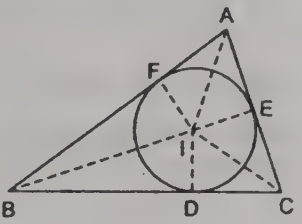
$$R = \frac{abc}{4S},$$

giving the radius of the circumcircle in terms of the sides.

202. *To find the value of r, the radius of the incircle of the triangle ABC.*

Bisect the two angles B and C by the two lines BI and CI meeting in I.

By Geometry, I is the centre of the incircle. Join IA, and draw ID, IE, and IF perpendicular to the three sides.



Then $ID = IE = IF = r$.

We have

$$\text{area of } \Delta IBC = \tfrac{1}{2}ID \,.\, BC = \tfrac{1}{2}r \,.\, a,$$

$$\text{area of } \Delta ICA = \tfrac{1}{2}IE \,.\, CA = \tfrac{1}{2}r \,.\, b,$$

and $$\text{area of } \Delta IAB = \tfrac{1}{2}IF \,.\, AB = \tfrac{1}{2}r \,.\, c,$$

Hence, by addition, we have

$$\tfrac{1}{2}r \,.\, a + \tfrac{1}{2}r \,.\, b + \tfrac{1}{2}r \,.\, c = \text{sum of the areas of the triangles } IBC, ICA, \text{ and } IAB$$

$$= \text{area of the } \Delta ABC,$$

$$\text{i.e. } r\frac{a+b+c}{2} = S,$$

so that $$r \,.\, s = S.$$

$$\therefore \quad r = \frac{S}{s}.$$

203. *Since the angles IBD and IDB are respectively equal to the angles IBF and IFB, the two triangles IDB and IFB are equal in all respects.*

Hence $BD = BF$, so that $2BD = BD + BF$.

So also $AE = AF$, so that $2AE = AE + AF$,

and $CE = CD$, so that $2CE = CE + CD$.

Hence, by addition, we have

$$2BD + 2AE + 2CE$$

$$= (BD + CD) + (CE + AE) + (AF + BF),$$

i.e. $$2BD + 2AC = BC + CA + AB.$$

$$\therefore 2BD + 2b = a + b + c = 2s.$$

Hence $$BD = s - b - BF;$$

so $$CE = s - c = CD,$$

and $$AF = s - a = AE.$$

Now $\frac{ID}{BD} = \tan IBD = \tan \frac{B}{2}$.

$\therefore r = ID = BD \tan \frac{B}{2} = (s - b) \tan \frac{B}{2}$.

So $r = IE = CE \tan ICE = (s - c) \tan \frac{C}{2}$,

and also $r = IF = AF \tan IAF = (s - a) \tan \frac{A}{2}$.

Hence $\mathbf{r = (s - a) \tan \frac{A}{2} = (s - b) \tan \frac{B}{2} = (s - c) \tan \frac{C}{2}}$.

204. *A third value for r may be found as follows :*

we have $a = BD + CD = ID \cot IBD + ID \cot ICD$

$$= r \cot \frac{B}{2} + r \cot \frac{C}{2}$$

$$= r \left[\frac{\cos \frac{B}{2}}{\sin \frac{B}{2}} + \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} \right]$$

$$\therefore a \sin \frac{B}{2} \sin \frac{C}{2} = r \left[\sin \frac{C}{2} \cos \frac{B}{2} + \cos \frac{C}{2} \sin \frac{B}{2} \right]$$

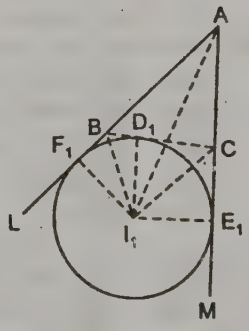
$$= r \sin \left(\frac{B}{2} + \frac{C}{2} \right) = r \sin \left[90° - \frac{A}{2} \right] = r \cos \frac{A}{2}.$$

$$\therefore \quad r = a \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}.$$

Cor. Since $a = 2R \sin A = 4R \sin \frac{A}{2} \cos \frac{A}{2}$,

we have $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

205. *To find the value of r_1, the radius of the escribed circle opposite the angle A of the triangle ABC.*



Produce AB and AC to L and M.

Bisect the angles CBL and BCM by the lines BI_1 and CI_1, and let these lines meet in I_1.

Draw I_1D_1, I_1E_1, and I_1F_1 perpendicular to the three sides respectively.

The two triangles I_1D_1B and I_1F_1B are equal in all respects, so that

$$I_1F_1 = I_1D_1.$$

Similarly $I_1E_1 = I_1D_1.$

The three perpendiculars I_1D_1, I_1E_1, and I_1F_1 being equal, the point I_1 is the centre of the required circle.

Now the area ABI_1C is equal to the sum of the triangles ABC and I_1BC; it is also equal to the sum of the triangles I_1BA and I_1CA.

Hence

$$\Delta ABC + \Delta I_1BC = \Delta I_1CA + \Delta I_1AB.$$

$$\therefore \quad S + \tfrac{1}{2}I_1D_1 \,.\, BC = \tfrac{1}{2}I_1E_1 \,.\, CA + \tfrac{1}{2}I_1F_1 \,.\, AB,$$

i.e. $$S + \tfrac{1}{2}r_1 \,.\, a = \tfrac{1}{2}r_1 \,.\, b + \tfrac{1}{2}r_1 \,.\, c.$$

$$\therefore \quad S = r_1\left[\frac{b+c-a}{2}\right] = r_1\left[\frac{b+c+a}{2} - a\right] = r_1(s-a).$$

$$\therefore \quad \mathbf{r_1 = \frac{S}{s-a}}.$$

Similarly it can be shewn that

$$\mathbf{r_2 = \frac{S}{s-b}}, \qquad \text{and} \qquad \mathbf{r_3 = \frac{S}{s-c}}.$$

206. Since AE_1 and AF_1 are tangents, we have, as in Art. 203, $AE_1 = AF_1$.

Similarly, $BF_1 = BD_1$, and $CE_1 = CD_1$.

$$\therefore \quad 2AE_1 = AE_1 + AF_1 = AB + BF_1 + AC + CE_1$$
$$= AB + BD_1 + AC + CD_1 = AB + BC + CA = 2s.$$

$$\therefore \quad AE_1 = s = AF_1.$$

Also, $BD_1 = BF_1 = AF_1 - AB = s - c,$

and $CD_1 = CE_1 = AE_1 - AC = s - b.$

$$\therefore \quad I_1E_1 = AE_1 \tan I_1AE_1,$$

i.e. $$\mathbf{r_1 = s \tan \frac{A}{2}}.$$

207. *A third value may be obtained for r_1 in terms of a and the angles, B and C.*

For, since I_1C bisects the angle BCE_1, we have

$$\angle I_1CD_1 = \tfrac{1}{2}(180° - C) = 90° - \frac{C}{2}.$$

So $$\angle I_1BD_1 = 90° - \frac{B}{2}.$$

$$\therefore a = BC = BD_1 + D_1C = I_1D_1 \cot I_1BD_1 + I_1D_1 \cot I_1CD_1$$

$$= r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)$$

$$= r_1 \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right)$$

$$\therefore a \cos \frac{B}{2} \cos \frac{C}{2} = r_1 \left(\sin \frac{B}{2} \cos \frac{C}{2} + \cos \frac{B}{2} \sin \frac{C}{2} \right)$$

$$= r_1 \sin \left(\frac{B}{2} + \frac{C}{2} \right) = r_1 \sin \left(90° - \frac{A}{2} \right) = r_1 \cos \frac{A}{2}.$$

$$\therefore \quad r_1 = a \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}.$$

Cor. Since $a = 2R \sin A = 4R \sin \frac{A}{2} \cos \frac{A}{2}$,

we have $r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

EXAMPLES. XXXVI.

1. In a triangle whose sides are 18, 24, and 30 cm respectively, prove that the circumradius, the inradius, and the radii of the three escribed circles are respectively 15, 6, 12, 18, and 36 cm.
2. The sides of a triangle are 13, 14, and 15 cm; prove that
   (1) $R = 8 \cdot 125$ cm, (2) $r = 4$ cm, (3) $r_1 = 10 \cdot 5$ cm,
   (4) $r_2 = 12$ cm, and (5) $r_3 = 14$ cm.
3. In a triangle ABC if $a = 13, b = 4$, and $\cos C = -\frac{5}{13}$, find R, r, r_1, r_2, and r_3.
4. In the ambiguous case of the solution of triangles prove that the circumcircles of the two triangles are equal.

Prove that

5. $r_1 (s - a) = r_2 (s - b) = r_3 (s - c) = rs = S$.
6. $\frac{rr_1}{r_2 r_3} = \tan^2 \frac{A}{2}$.
7. $rr_1 r_2 r_3 = S^2$.
8. $r_1 r_2 r_3 = r^3 \cot^2 \frac{A}{2} \cot^2 \frac{B}{2} \cot^2 \frac{C}{2}$.
9. $rr_1 \cot \frac{A}{2} = S$.
10. $r_2 r_3 + r_3 r_1 + r_1 r_3 = s^2$.
11. $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r} = 0$.
12. $a (rr_1 + r_2 r_3) = b (rr_2 + r_3 r_1) = c (rr_3 + r_1 r_2)$.
13. $(r_1 + r_2) \tan \frac{C}{2} = (r_3 - r) \cot \frac{C}{2} = c$.

14. $S = 2R^2 \sin A \sin B \sin C$.

15. $4R \sin A \sin B \sin C = a \cos A + b \cos B + c \cos C$.

16. $S = 4Rr \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$.

17. $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{S^2}$. **18.** $r_1 + r_2 + r_3 - r = 4R$.

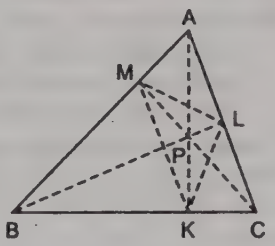
19. $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$

20. $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}$. **21.** $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$.

22. $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$.

208. Orthocentre and pedal triangle of any triangle.

Let ABC be any triangle, and let AK, BL, and CM be the perpendiculars from A, B, and C upon the opposite sides of the triangle. It can be easily shewn, as in most books on Geometry, that these three perpendiculars meet in a common point P. This point P is called the **orthocentre** of the triangle. The triangle KLM, which is formed by joining the feet of these perpendiculars, is called the **pedal triangle** of ABC.



209. Distances of the orthocentre from the angular points of the triangle.

We have $PK = KB \tan PBK = KB \tan (90° - C)$

$$= AB \cos B \cot C = \frac{c}{\sin C} \cos B \cos C$$

$$= 2R \cos B \cos C. \qquad \text{(Art. 200.)}$$

Again $AP = AL \sec KAC$

$$= c \cos A \operatorname{cosec} C$$

$$= \frac{c}{\sin C} \cos A$$

$$= 2R \cos A. \qquad \text{(Art. 200.)}$$

So $BP = 2R \cos B$, and $CP = 2R \cos C$.

The distances of the orthocentre from the angular points are therefore $2R \cos A$, $2R \cos B$, and $2R \cos C$; its distances from the sides are $2R \cos B \cos C$, $2R \cos C \cos A$, and $2R \cos A \cos B$.

210. *To find the sides and angles of the pedal triangle.*

Since the angles PKC and PLC are right angles, the points P, L, C, and K lie on a circle.

$$\therefore \quad \angle PKL = \angle PCL$$

$$= 90° - A.$$

Similarly, P, K, B, and M lie on a circle, and therefore

$$\angle PKM = \angle PBM$$

$$= 90° - A.$$

Hence $\angle MKL = 180° - 2A$
$= \text{the supplement of } 2A.$

So $\angle KLM = 180° - 2B,$

and $\angle LMK = 180° - 2C.$

Again, from the triangle ALM, we have

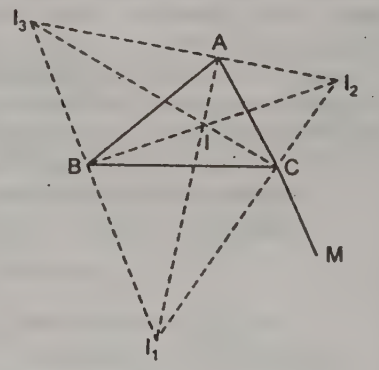
$$\frac{LM}{\sin A} = \frac{AL}{\sin AML} = \frac{AB \cos A}{\cos PML}$$
$$= \frac{c \cos A}{\cos PAL} = \frac{c \cos A}{\sin C}.$$

$$\therefore \quad LM = \frac{c}{\sin C} \sin A \cos A$$
$$= a \cos A. \qquad \text{(Art. 163.)}$$

So $MK = b \cos B$, and $KL = c \cos C$.

The sides of the pedal triangle are therefore $a \cos A, b \cos B$, and $c \cos C$; also its angles are the supplements of twice the angles of the triangle.

211. Let I be the centre of the incircle and I_1, I_2, and I_3 the centres of the escribed circles which are opposite to A, B, and C respectivley. As in Arts. 202 and 205, IC bisects the angle ACB, and I_1C bisects the angle BCM.



$$\therefore \quad \angle ICI_1 = \angle ICB + \angle I_1CB$$
$$= \tfrac{1}{2} \angle ACB + \tfrac{1}{2} \angle MCB$$
$$= \tfrac{1}{2} [\angle ACB + \angle MCB]$$
$$= \tfrac{1}{2} . 180° = \text{a right angle.}$$

Similarly, $\angle ICI_2$ is a right angle.

Hence I_1CI_2 is a straight line to which IC is perpendicular.

So I_2AI_3 is a straight line to which IA is perpendicular, and I_3BI_1 is a straight line to which IB is perpendicular.

Also, since IA and I_1A both bisect the angle BAC, the three points A, I, and I_1 are in a straight line. Similarly BII_2 and CII_3 are straight lines. Hence $I_1I_2I_3$ is a triangle, which is such that A, B, and C are the feet of the perpendiculars drawn from its vertices upon the opposite sides, and such that I is the intersection of these perpendiculars,

i.e. ABC is its pedal triangle and I is its orthocentre.

The triangle $I_1I_2I_3$ is often called the excentric triangle.

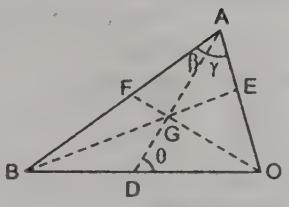
212. Centroid and medians of any triangle.

If ABC be any triangle, and D, E, and F respectively the middle points of BC, CA, and AB, the lines AD, BE, and CF are called the **medians** of the triangle.

It is shewn in most books on Geometry that the medians meet in a common point G, such that

$$AG = \tfrac{2}{3}AD, BG = \tfrac{2}{3}BE,$$

and $$CG = \tfrac{2}{3}CF.$$



This point G is called the **centroid** of the triangle.

213. *Length of the medians.* We have, by Art. 164,

$$AD^2 = AC^2 + CD^2 - 2AC \cdot CD \cos C$$

$$= b^2 + \frac{a^2}{4} - ab \cos C,$$

and $$c^2 = b^2 + a^2 - 2ab \cos C.$$

Hence $$2AD^2 - c^2 = b^2 - \frac{a^2}{2},$$

so that $$AD = \tfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}.$$

Hence also $$AD = \tfrac{1}{2}\sqrt{b^2 + c^2 + 2bc \cos A}.$$ (Art. 164.)

So also

$$BE = \tfrac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}, \text{ and } CF = \tfrac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}.$$

214. *Angles that the median AD makes with the sides.*

If the $\angle BAD = \beta$, and $\angle CAD = \gamma$, we have

$$\frac{\sin\gamma}{\sin C} = \frac{DC}{AD} = \frac{a}{2x}.$$

$$\therefore \quad \sin\gamma = \frac{a \sin C}{2x} = \frac{a \sin C}{\sqrt{2b^2 + 2c^2 - a^2}}.$$

Similarly, $$\sin\beta = \frac{a \sin B}{\sqrt{2b^2 + 2c^2 - a^2}}.$$

Again, if the $\angle ADC$ be θ, we have

$$\frac{\sin\theta}{\sin C} = \frac{AC}{AD} = \frac{b}{x}.$$

$$\therefore \quad \sin\theta = \frac{b \sin C}{x} = \frac{2b \sin C}{\sqrt{2b^2 + 2c^2 - a^2}}.$$

The angles that AD makes with the sides are therefore found.

215. *The centroid lies on the line joining the circumcentre to the orthocentre.*

Let O and P be the circumcentre and orthocentre respectively. Draw OD and PK perpendicular to BC.

Let AD and OP meet in G.

The triangles OGD and PGA are clearly equiangular.

Also, by Art. 200,

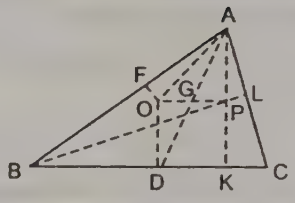
$$OD = R\cos A$$

and, by Art. 209,

$$AP = 2R\cos A.$$

Hence, by similar triangles,

$$\frac{AG}{GD} = \frac{AP}{OD} = 2.$$



The point G is therefore the centroid of the triangle.

Also, by the same proposition,

$$\frac{OG}{GP} = \frac{OD}{AP} = \frac{1}{2}.$$

The centroid therefore lies on the line joining the circumcentre to the orthocentre, and divides it in the ratio 1 : 2.

It may be shewn by Geometry that the centre of the nine-point circle (which passes through the feet of the perpendiculars, the middle points of the sides, and the middle points of the lines joining the angular points to the orthocentre) lines on OP and bisects it.

The circumcentre, the centroid, the centre of the nine-point circle, and the orthocentre therefore all lie on a straight line.

216. *Distance between the circumcentre and the orthocentre.*

If OF be perpendicular to AB, we have

$$\angle OAF = 90° - \angle AOF = 90° - C.$$

Also $\angle PAL = 90° - C.$

$$\therefore \quad \angle OAP = A - \angle OAF - \angle PAL$$
$$= A - 2(90° - C) = A + 2C - 180°$$
$$= A + 2C - (A + B + C) = C - B.$$

Also $OA = R$, and, by Art. 209.

$$PA = 2R\cos A.$$

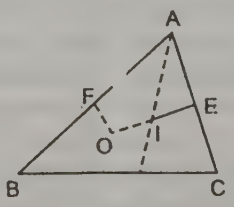
$$\therefore \quad OP^2 = OA^2 + PA^2 - 2OA \,.\, PA\cos OAP$$
$$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$$
$$= R^2 + 4R^2\cos A[\cos A - \cos(C - B)]$$
$$= R^2 - 4R^2\cos A[\cos(B + C) + \cos(C - B)] \qquad \text{(Art. 72)},$$
$$= R^2 - 8R^2\cos A\cos B\cos C.$$

$$\therefore OP = R\sqrt{1 - 8\cos A\cos B\cos C}.$$

217. *To find the distance between the circumcentre and the incentre.*

Let O be the circumcentre, and OF be perpendicular to AB.

Let I be the incentre, and IE be perpendicular to AC.



Then, as in the previous article,

$$\angle OAF = 90° - C.$$

$$\therefore \quad \angle OAI = \angle IAF - \angle OAF$$

$$= \frac{A}{2} - (90° - C) = \frac{A}{2} + C - \frac{A+B+C}{2} = \frac{C-B}{2}.$$

Also

$$AI = \frac{IE}{\sin\frac{A}{2}} = \frac{r}{\sin\frac{A}{2}} = 4R\sin\frac{B}{2}\sin\frac{C}{2} \qquad \text{(Art. 204. Cor.)}$$

$\therefore$ $OI^2 = OA^2 + AI^2 - 2OA \,.\, AI\cos OAI$

$$= R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$$

$\therefore$ $$\frac{OI^2}{R^2} = 1 + 16\sin^2\frac{B}{2}\sin^2\frac{C}{2}$$

$$- 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B+C}{2}$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2} \text{ (Art. 69)} \qquad \ldots(1)$$

$\therefore$ $$OI = R\sqrt{1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}}.$$

Also (1) may be written

$$OI^2 = R^2 - 2R \times 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= R^2 - 2Rr. \qquad \text{(Ar. 204. Cor.)}$$

In a similar manner it may be shewn that, if I_1 be the centre of the escribed circle opposite the angle A, we shall have

$$OI_1 = R\sqrt{1 + 8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}.$$

and hence $OI_1^2 = R^2 + 2Rr_1$. (Art. 207. Cor.)

Aliter. Let OI be produced to meet the circumcircle of the triangle in S and T, and let AII_1 meet it in H.

By Geometry, we have

$$SI \,.\, IT = AI \,.\, IH \qquad \ldots(2)$$

But $SI \,.\, IT = (R + OI)(R - OI) = R^2 - OI^2$.

Also $\angle HIC = \angle ICA + \angle IAC = \angle ICB + \angle HAB$

$$= \angle ICB + \angle HCB$$

$$= \angle HCI.$$

$$\therefore \qquad III = HC = 2R \sin\frac{A}{2}. \quad \text{(Art. 200).}$$

Also $$AI = \frac{IE}{\sin\frac{A}{2}} = \frac{r}{\sin\frac{A}{2}}.$$

Substituting in (2), we have

$$R^2 - OI^2 = 2Rr,$$

i.e. $$OI^2 = R^2 - 2Rr.$$

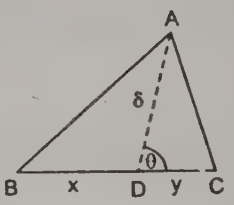
Similarly, we can shew that $I_1H = I_1C$, and hence that

$$I_1O^2 - R^2 = I_1H \,.\, I_1A = 2Rr_1,$$

i.e. $$I_1O^2 = R^2 + 2Rr_1.$$

218. Bisectors of the angles.

If AD bisect the angle A and divide the base into portions x and y, we have, by Geometry,



$$\frac{x}{y} = \frac{AB}{AC} = \frac{c}{b}.$$

$$\therefore \qquad \frac{x}{c} = \frac{y}{b} = \frac{x+y}{b+c} = \frac{a}{b+c} \qquad \ldots(1),$$

giving x and y.

Also, if δ be the length of AD and θ the angle it makes with BC, we have

$$\Delta ABD + \Delta ACD = \Delta ABC.$$

$$\therefore \qquad \frac{1}{2}c\delta \sin\frac{A}{2} + \frac{1}{2}b\delta \sin\frac{A}{2} = \frac{1}{2}bc \sin A,$$

i.e. $$\delta = \frac{bc}{b+c} \frac{\sin A}{\sin\frac{A}{2}} = \frac{2bc}{b+c} \cos\frac{A}{2} \qquad \ldots(2)$$

Also $$\theta = \angle DAB + B = \frac{A}{2} + B \qquad \ldots(3)$$

We thus have the length of the bisector and its inclination to BC.

EXAMPLES. XXXVII.

If I, I_1, I_2, and I_3 be respectively the centres of the incircle and the three escribed circles of a triangle ABC, prove that

1. $AI = r \operatorname{cosec}\frac{A}{2}$.

2. $IA \,.\, IB \,.\, IC = abc \tan\frac{A}{2} \tan\frac{B}{2} \tan\frac{C}{2}$.

3. $AI_1 = r_1 \operatorname{cosec}\frac{A}{2}$.

4. $II_1 = a \sec\frac{A}{2}$.

5. $I_2I_3 = a \operatorname{cosec} \frac{A}{2}$.

6. $II_1 . II_2 . II_3 = 16R^2r$.

7. $I_2I_3^2 = 4R(r_2 + r_3)$.

8. $\angle I_3I_1I_2 = \frac{B+C}{2}$.

9. $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$.

10. Area of $\Delta I_1I_2I_3 = 8R^2 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} = \frac{abc}{2r}$.

11. $\frac{II_1 . I_2I_3}{\sin A} = \frac{II_2 . I_3I_1}{\sin B} = \frac{II_3 . I_1I_2}{\sin C}$.

If I, O, and P be respectively the incentre, circumcentre, and orthocentre, and G the centroid of the triangle ABC, prove that

12. $IO^2 = R^2(3 - 2\cos A - 2\cos B - 2\cos C)$.

13. $IP^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$.

14. $OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$.

15. Area of $\Delta IOP = 2R^2 \sin\frac{B-C}{2} \sin\frac{C-A}{2} \sin\frac{A-B}{2}$.

16. Area of $\Delta IPG = \frac{4}{3}R^2 \sin\frac{B-C}{2} \sin\frac{C-A}{2} \sin\frac{A-B}{2}$.

17. Prove that the distance of the centre of the nine-point circle from the angle A is $\frac{R}{2}\sqrt{1 + 8\cos A \sin B \sin C}$.

18. DEF is the pedal triangle of ABC; prove that
(1) its area is $2S \cos A \cos B \cos C$,
(2) the radius of its circumcircle is $\frac{R}{2}$,
and (3) the radius of its incircle is $2R \cos A \cos B \cos C$.

19. $O_1O_2O_3$ is the triangle formed by the centres of the escribed circles of the triangle ABC; prove that
(1) its sides are $4R \cos\frac{A}{2}$, $4R \cos\frac{B}{2}$, and $4R \cos\frac{C}{2}$,
(2) its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$, and $\frac{\pi}{2} - \frac{C}{2}$,
and (3) its area is $2Rs$.

20. DEF is the triangle formed by joining the points of contact of the incircle with the sides of the triangle ABC; prove that
(1) its sides are $2r \cos\frac{A}{2}$, $2r \cos\frac{B}{2}$, and $2r \cos\frac{C}{2}$,
(2) its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$, and $\frac{\pi}{2} - \frac{C}{2}$,
and (3) its area is $\frac{2S^3}{abcs}$, *i.e.* $\frac{1}{2}\frac{r}{R}S$.

21. D, E, and F are the middle points of the sides of the triangle ABC; prove that the centroid of the triangle DEF is the same as that of ABC, and that its orthocentre is the circumcentre of ABC.

In any triangle ABC, prove that

22. The perpendicular from A divides BC into portions which are proportional to the cotangents of the adjacent angles, and that it divides the angle A into portions whose cosines are inversely proportional to the adjacent sides.

23. The median through A divides it into angles whose cotangents are $2\cot A + \cot C$ and $2\cot A + \cot B$, and makes with the base an angle whose contangent is $\frac{1}{2}(\cot C - \cot B)$.

24. The distance between the middle point of BC and the foot of the perpendicular from A is $\frac{b^2 - c^2}{2a}$.

25. O is the orthocentre of a triangle ABC; prove that the radii of the circles circumscribing the triangles BOC, COA, AOB, and ABC are all equal.

26. AD, BE, and CF are the perpendiculars from the angular points of a triangle ABC upon the opposite sides; prove that the diameters of the circumcircles of the triangles AEF, BDF, and CDE are respectively $a\cot A, b\cot B$, and $c\cot C$, and that the perimeters of the triangles DEF and ABC are in the ratio $r : R$.

27. Prove that the product of the distances of the incentre from the angular points of a triangle is $4Rr^2$.

28. The triangle DEF circumscribes the three escribed circles of the triangle ABC; prove that

$$\frac{EF}{a\cos A} = \frac{FD}{b\cos B} = \frac{DE}{c\cos C}.$$

29. If a circle be drawn touching the inscribed and circumscribed circles of a triangle and the side BC externally, prove that its radius is

$$\frac{\Delta}{a}\tan^2\frac{A}{2}.$$

30. If a, b, and c be the radii of three circles which touch one another externally, and r_1 and r_2 be the radii of the two circles that can be drawn to touch these three, prove that

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$

31. If Δ_0 be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides of the given triangle, whose area is Δ, and Δ_1, Δ_2, and Δ_3 the corresponding areas for the escribed circles, prove that

$$\Delta_1 + \Delta_2 + \Delta_3 - \Delta_0 = 2\Delta.$$

32. If the bisectors of the angles of a triangle ABC meet the opposite sides in A', B', and C', prove that the ratio of the areas of the triangles $A'B'C'$ and ABC is

$$2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} : \cos \frac{A-B}{2} \cos \frac{B-C}{2} \cos \frac{C-A}{2}.$$

33. Through the angular points of a triangle are drawn straight lines which make the same angle a with the opposite sides of the triangle; prove that the area of the triangle formed by them is to the area of the original triangle as $4 \cos^2 a : 1$.

34. Two circles, of radii a and b, cut each other at an angle θ. Prove that the length of the common chord is

$$\frac{2ab \sin \theta}{\sqrt{a^2 + b^2 + 2ab \cos \theta}}.$$

35. Three equal circles touch one another; find the radius of the circle which touches all three.

36. Three circles, whose radii are a, b, and c, touch one another externally and the tangents at their points of contact meet in a point; prove that the distance of this point from either of their points of contact is

$$\left(\frac{abc}{a+b+c}\right)^{1/2}$$

37. In the sides BC, CA, AB are taken three points A', B', C' such that

$$BA' : A'C = CB' : B'A = AC' : C'B = m : n;$$

prove that if AA', BB', and CC' be joined they will form by their intersections a triangle whose area is to that of the triangle ABC as $(m-n)^2 : m^2 + mn + n^2$.

38. The circle inscribed in the triangle ABC touches the sides BC, CA, and AB in the points A_1, B_1, and C_1 respectively; similarly the circle inscribed in the triangle $A_1B_1C_1$ touches the sides in A_2, B_2, C_2 respectively, and so on ; if $A_nB_nC_n$ be the nth triangle so formed, prove that its angles are

$$\frac{\pi}{3} + (-2)^{-n}\left(A - \frac{\pi}{3}\right), \frac{\pi}{3} + (-2)^{-n}\left(B - \frac{\pi}{3}\right),$$

and $\frac{\pi}{3} + (-2)^{-n}\left(C - \frac{\pi}{3}\right)$.

Hence prove that the triangle so formed is ultimately equilateral.

39. $A_1B_1C_1$ is the triangle formed by joining the feet of the perpendiculars drawn from ABC upon the opposite sides; in like manner $A_2B_2C_2$ is the triangle obtained by joining the feet of the perpendiculars from A_1, B_1 and C_1 on the opposite sides, and so on. Find the values of the angles A_n, B_n, and C_n in the nth of these triangles.

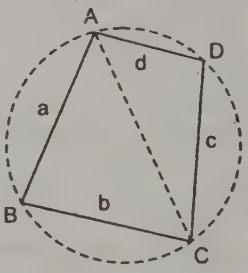
40. The legs of a tripod are each 10 cm in length, and their points of contact with a horizontal table on which the tripod stands from a triangle whose sides are 7, 8 and 9 cm in length. Find the inclination of the legs to the horizontal, and the height of the apex.

CHAPTER XVI

ON QUADRILATERALS AND REGULAR POLYGONS

219. *To find the area of a quadrilateral which is inscribable in a circle.*

Let $ABCD$ be the quadrilateral, the sides being a, b, c, and d as marked in the figure.



The area of the quadrilateral

$$= \text{area of } \Delta ABC + \text{area of } \Delta ADC$$

$$= \tfrac{1}{2}ab \sin B + \tfrac{1}{2}cd \sin D \text{ (Art. 198)}$$

$$= \tfrac{1}{2}(ab + cd) \sin B,$$

since, by Geometry,

$$\angle B = 180° - \angle D,$$

and therefore

$$\sin B = \sin D.$$

We have to express $\sin B$ in terms the sides.

We have

$$a^2 + b^2 - 2ab \cos B = AC^2 = c^2 + d^2 - 2cd \cos D.$$

But $\cos D = \cos (180° - B) = -\cos B.$

Hence

$$a^2 + b^2 - 2ab \cos B = c^2 + d^2 + 2cd \cos B,$$

so that $\cos B = \dfrac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$

Hence

$$\sin^2 B = 1 - \cos^2 B = 1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{\{2(ab + cd)\}^2}$$

$$= \frac{\{2(ab + cd)\}^2 - \{a^2 + b^2 - c^2 - d^2\}^2}{4(ab + cd)^2}$$

$$= \frac{\{2(ab + cd) + (a^2 + b^2 - c^2 - d^2)\}\{2(ab+cd) - (a^2+b^2-c^2-d^2)\}}{4(ab + cd)^2}$$

$$= \frac{\{(a^2 + 2ab + b^2) - (c^2 - 2cd + d^2)\}\{(c^2 + 2cd + d^2) - (a^2 + b^2 - 2ab)\}}{4(ab + cd)^2}$$

$$= \frac{\{(a + b)^2 - (c - d)^2\}\{(c + d)^2 - (a - b)^2\}}{4(ab + cd)^2}$$

$$= \frac{\{(a+b+c-d)(a+b-c+d)\}\{(c+d+a-b)(c+d-a+b)\}}{4(ab + cd)^2}.$$

Let

$$a + b + c + d = 2s,$$

so that

$$a + b + c - d = (a + b + c + d) - 2d = 2(s - d),$$
$$a + b - c + d = 2(s - c),$$
$$a - b + c + d = 2(s - b),$$

and $$-a + b + c + d = 2(s - a).$$

Hence

$$\sin^2 B = \frac{2(s - d) \times 2(s - c) \times 2(s - b) \times 2(s - a)}{4(ab + cd)^2},$$

so that

$$(ab + cd) \sin B = 2\sqrt{(s - a)(s - b)(s - c)(s - d)}.$$

Hence the area of the quadrilateral

$$= \tfrac{1}{2}(ab + cd) \sin B = \sqrt{(s - a)(s - b)(s - c)(s - d)}.$$

220. *Since* $\cos B = \dfrac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$,

we have $AC^2 = a^2 + b^2 - 2ab \cos B$

$$= a^2 + b^2 - ab \frac{a^2 + b^2 - c^2 - d^2}{ab + cd}$$

$$= \frac{(a^2 + b^2)cd + ab(c^2 + d^2)}{ab + cd}$$

$$= \frac{(ac + bd)(ad + bc)}{ab + cd}.$$

Similarly it could be proved that

$$BD^2 = \frac{(ab + cd)(ac + bd)}{ad + bc}.$$

We thus have the lengths of the diagonals of the quadrilateral. It follows by multiplication that

$$AC^2 . BD^2 = (ac + bd)^2,$$

i.e. $$AC . BD = AB/CD + BC . AD.$$

This is a well-known proposition in Geometry.

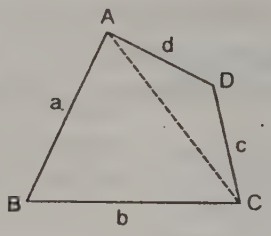
Again, the radius of the circle circumscribing the quadrilateral

$$= \frac{1}{2}\frac{AC}{\sin B}$$

$$= \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \div 4\sqrt{\frac{(s-a)(s-b)(s-c)(s-d)}{(ab+cd)^2}}$$

$$= \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}.$$

221. If we have any quadrilateral, not necessarily inscribable in a circle, we can express its area in terms of its sides and the sum of any two opposite angles.



For let the sum of the two angles B and D be denoted by 2α, and denote the area of the quadrilateral by Δ.

Then Δ = area of ABC + area of ACD

$$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D,$$

so that

$$4\Delta = 2ab\sin B + 2cd\sin D \qquad \ldots(1)$$

Also $a^2 + b^2 - 2ab\cos B = c^2 + d^2 - 2cd\cos D$,

so that

$$a^2 + b^2 - c^2 - d^2 = 2ab\cos B - 2cd\cos D \qquad \ldots(2)$$

Squaring (1) and (2) and adding, we have

$$16\Delta^2 + (a^2+b^2-c^2-d^2)^2 = 4a^2b^2 + 4c^2d^2 - 8abcd(\cos B\cos D - \sin B\sin D)$$

$$= 4a^2b^2 + 4c^2d^2 - 8abcd\cos(B+D)$$

$$= 4a^2b^2 + 4c^2d^2 - 8abcd\cos 2\alpha$$

$$= 4a^2b^2 + 4c^2d^2 - 8abcd(2\cos^2\alpha - 1)$$

$$= 4(ab+cd)^2 - 16abcd\cos^2\alpha,$$

so that

$$16\Delta^2 = 4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2 - 16abcd\cos^2\alpha \qquad \ldots(3)$$

But, as in Art. 219, we have

$$4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2 = 2(s-a).2(s-b).2(s-c).2(s-d)$$

$$= 16(s-a)(s-b)(s-c)(s-d)$$

Hence (3) becomes

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha,$$

giving the required area.

Cor. 1. If d be zero, the quadrilateral becomes a triangle, and the formula above becomes that of 198.

Cor. 2. If the sides of the quadrilateral be given in length, we know a, b, c, d and therefore s. The area Δ is hence greatest when $abcd \cos^2 \alpha$ is least, that is when $\cos^2 \alpha$ is zero, and then $\alpha = 90°$. In this case the sum of two opposite angles of the quadrilateral is 180° and the figure inscribable in a circle.

The quadrilateral, whose sides are given, has therefore the greatest area when it can be inscribed in a circle.

222. Ex. *Find the area of a quadrilateral which can have a circle inscribed in it.*

If the quadrilateral $ABCD$ can have a circle inscribed in it so as to touch the sides AB, BC, CD, and DA in the points P, Q, R, and S, we should have

$$AP = AS, BP = BQ, CQ = CR, \text{ and } DR = DS.$$

$$\therefore \quad AP + BP + CR + DR = AS + BQ + CQ + DS,$$

$$i.e. \quad AB + CD = BC + DA,$$

$$i.e. \quad a + c = b + d.$$

Hence, $$s = \frac{a+b+c+d}{2} = a + c = b + d$$

$$\therefore \quad s - a = c, s - b = d, s - c = a, \text{ and } s - d = b.$$

The formula of the previous article therefore gives in this case

$$\Delta^2 = abcd - abcd \cos^2 a = abcd \sin^2 a,$$

i.e. the area required $= \sqrt{abcd} \sin a$.

If in addition the quadrilateral be also inscribable in a circle, we have

$$2a = 180°, \text{ so that } \sin a = \sin 90° = 1.$$

Hence the area of a quadrilateral which can be both inscribed in a circle and circumscribed about another circle is $\sqrt{abcd}$.

EXAMPLES. XXXVIII.

1. Find the area of a quadrilateral, which can be inscribed in a circle, whose sides are
   (1) 3, 5, 7 and 9 cm;
   and (2) 7, 10, 5, and 2 cm.
2. The sides of a quadrilateral are respectively 3, 4, 5, and 6 cm, and the sum of a pair of opposite angles is 120°; prove that the area of the quadrilateral is $3\sqrt{30}$ square cm.
3. The sides of a quadrilateral which can be inscribed in a circle are 3, 3, 4, and 4 cm; find the radii of the incircle and circumcircle.
4. Prove that the area of any quadrilateral is one-half the product of the two diagonals and the sine of the angle between them.
5. If a quadrilateral can be inscribed in one circle and circumscribed about another circle, prove that its area is $\sqrt{abcd}$, and that the radius of the latter circle is
$$\frac{2\sqrt{abcd}}{a+b+c+d}.$$

6. A quadrilateral $ABCD$ is described about a circle; prove that

$$AB \sin\frac{A}{2}\sin\frac{B}{2} = CD \sin\frac{C}{2}\sin\frac{D}{2}.$$

7. a, b, c, and d are the sides of a quadrilateral taken in order, and a is the angle between the diagonals opposite to b or d; prove that the area of the quadrilateral is

$$\frac{1}{4}(a^2 - b^2 + c^2 - d^2)\tan a.$$

8. If a, b, c, and d be the sides and x and y the diagonals of a quadrilateral, prove that its area is

$$\frac{1}{4}[4x^2y^2 - (b^2 + d^2 - a^2 - c^2)^2]^{1/2}.$$

9. If a quadrilateral can be inscribed in a circle, prove that the angle between its diagonals is

$$\sin^{-1}[2\sqrt{(s-a(s-b)(s-c)(s-d)} \div (ac + bd)]$$

If the same quadrilateral can also be circumscribed about a circle, prove that this angle is then

$$\cos^{-1}\frac{ac - bd}{ac + bd}.$$

10. The sides of a quadrilateral are divided in order in the ratio $m : n$, and a new quadrilateral is formed by joining the points of division; prove that its area is to the area of the original figure as $m^2 + n^2$ to $(m + n)^2$.

11. If $ABCD$ be a quadrilateral incribed in a circle, prove that

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}},$$

and that the product of the segments into which one diagonal is divided by the other diagonal is

$$\frac{abcd(ac + bd)}{(ab + cd)(ad + bc)}.$$

12. If a, b, c, and d be the sides of a quadrilateral, taken in order, prove that

$$d^2 = a^2 + b^2 + c^2 - 2ab\cos\alpha - 2bc\cos\beta - 2ca\cos\gamma,$$

where α, β and γ denote the angles between the sides a and b, b and c, and c and a respectively.

223. Regular Polygons. A regular polygon is a polygon which has all its sides equal and all its angles equal.

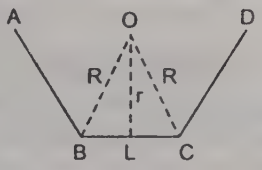
If the polygon have n angles we have, by Geometry, n times its angle + 4 right angle = twice as many right angles as the figure has sides = $2n$ right angles.

Hence each angle $= \frac{2n-4}{n}$ right angles $= \frac{2n-4}{n} \times \frac{\pi}{2}$ radians.

224. *Radii of the inscribed and circumscribing circles of a regular polygon.*

Let AB, BC, and CD be three successive sides of the polygon, and let n be the number of its sides.

Bisect the angles ABC and BCD by the lines BO and CO which meet in O, and draw OL perpendicular to BC.



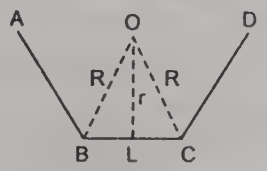
It is easily seen that O is the centre of both the incircle and the circumcircle of the polygon, and that BL equals LC.

Hence we have $OB = OC = R$, the radius of the circumcircle, and $OL = r$, the radius of the incircle.

The angle BOC is $\frac{1}{n}$th of the sum of all the angles subtended at O by the sides, i.e.

$$\angle BOC = \frac{4 \text{ right angles}}{n}$$

$$= \frac{2\pi}{n} \text{ radians.}$$



Hence

$$\angle BOL = \frac{1}{2} \angle BOC = \frac{\pi}{n}.$$

If a be a side of the polygon, we have

$$a = BC = 2BL = 2R \sin BOL = 2R \sin \frac{\pi}{n}.$$

$$\therefore \quad R = \frac{a}{2 \sin \frac{\pi}{n}} = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n} \qquad \ldots(1)$$

Again, $a = 2BL = 2OL \tan BOL = 2r \tan \frac{\pi}{n}.$

$$\therefore \quad r = \frac{a}{2 \tan \frac{\pi}{n}} = \frac{a}{2} \cot \frac{\pi}{n} \qquad \ldots(2)$$

225. Area of a Regular Polygon.

The area of the polygon is n times the area of the triangle BOC.

Hence the area of the polygon

$$= n \times \frac{1}{2} OL \,.\, BC = n \,.\, OL \,.\, BL = n \,.\, BL \cot LOB \,.\, BL$$

$$= n \cdot \frac{a^2}{4} \cot \frac{\pi}{n} \qquad \ldots(1)$$

an expression for the area in terms of the side.

Also the area

$$= n \cdot OL \cdot BL = n \cdot OL \cdot OL \tan BOL = nr^2 \tan \frac{\pi}{n} \quad \ldots(2)$$

Again, the area

$$= n \cdot OL \cdot BL = n \cdot OB \cos LOB \cdot OB \sin LOB$$

$$= nR^2 \cos \frac{\pi}{n} \sin \frac{\pi}{n} = \frac{n}{2} R^2 \sin \frac{2\pi}{n} \quad \ldots(3)$$

The formulae (2) and (3) give the area in terms of the radius of the inscribed and circumscribed circles.

226. Ex. *The length of each side of a regular dodecagon is* 20 *feet; find* (1) *the radius of its inscribed circle,* (2) *the radius of its circumscribing circle, and* (3) *its area.*

The angle subtended by a side at the centre of the polygon

$$= \frac{360°}{12} = 30°.$$

Hence we have $10 = r \tan 15° = R \sin 15°$.

$$\therefore \quad r = 10 \cot 15° = \frac{10}{2 - \sqrt{3}} \quad \text{(Art. 101)}$$

$$= 10 (2 + \sqrt{3}) = 37{\cdot}32\ldots\text{cm.}$$

$$\text{Also } R = \frac{10}{\sin 15°} = 10 \times \frac{2\sqrt{2}}{\sqrt{3} - 1} \quad \text{(Art. 106)}$$

$$= 10 \cdot \sqrt{2} (\sqrt{3} + 1) = 10 (\sqrt{6} + \sqrt{2})$$

$$= 10 (2{\cdot}4495\ldots + 1{\cdot}4142\ldots) = 38{\cdot}637 \ldots \text{ cm.}$$

Again, the area $= 12 \times r \times 10$ square feet

$$= 1200 (2 + \sqrt{3}) = 4478{\cdot}46\ldots\text{square cm.}$$

EXAMPLES. XXXIX.

1. Find, correct to ·01 of a cm, the length of the perimeter of a regular decagon which surrounds a circle of radius 12 cm. [tan 18° = ·32492.]
2. Find to 3 places of decimals the length of the side of a regular polygon of 12 sides which is circumscribed to a circle of unit radius.
3. Find the area of (1) a pentagon, (2) a hexagon, (3) an octagon, (4) a decagon and (5) a dodecagon, each being a regular figure of side 1 metre. [cot 18° = 3·07768...; cot 36° = 1·37638...]
4. Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each is 24 cm.
5. A square, whose side is 2 cm, has its corners cut away so as to form a regular octagon; find its area.
6. Compare the areas and perimeters of octagons which are respectively inscribed in and circumscribed to a given circle, and shew that the areas of the incribed hexagon and octagon are as $\sqrt{27}$ to $\sqrt{32}$.
7. Prove that the radius of the circle described about a regular pentagon is nearly $\frac{17}{20}$th of the side of the pentagon.
8. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as 2 : 3.

9. If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2 : \sqrt{5}$.
10. Prove that the sum of the radii of the circles, which are respectively inscribed in and circumscribed about a regular polygon of n sides, is $\frac{a}{2}\cot\frac{\pi}{2n}$, where a is a side of the polygon.
11. Of two regular polygons of n sides, one circumscribes and the other is inscribed in a given circle. Prove that the perimeters of the circumscribing polygon, the circle, and the inscribed polygon are in the ratio

$$\sec\frac{\pi}{n} : \frac{\pi}{n} : \operatorname{cosec}\frac{\pi}{n} : 1,$$

and that the areas of the polygons are in the ratio $\cos^2\frac{\pi}{n} : 1$.

12. Given that the area of a polygon of n sides circumscribed about a circle is to the area of the circumscribed polygon of $2n$ sides as $3 : 2$, find n.
13. Prove that the area of a regular polygon of $2n$ sides inscribed in a circle is a mean proportional between the areas of the regular inscribed and circumscribed polygons of n sides.
14. The area of a regular polygon of n sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as 3 is to 4. Find the value of n.
15. The interior angles of a polygon are in A.P.; the least angle is 120° and the common difference is 5°; find the number of sides.
16. There are two regular polygons, the number of sides in one being double the number in the other, and an angle of one polygon is to an angle of the other as 9 to 8; find the number of sides of each polygon.
17. Shew that there are eleven pairs of regular polygons such that the number of degrees in the angle of one is to the number in the angle of the other as $10 : 9$. Find the number of sides in each.
18. The side of a base of a square pyramid is a metre and its vertex is at a height of h metres above the centre of the base; if θ and ϕ be respectively the inclinations of any face to the base, and of any two faces to one another, prove that

$$\tan\theta\frac{2h}{a} \text{ and } \tan\frac{\phi}{2} = \sqrt{1 + \frac{a^2}{2h^2}}.$$

19. A pyramid stands on a regular hexagon as base. The perpendicular from the vertex of the pyramid on the base passes through the centre of the hexagon, and its length is equal to that of a side of the base. Find the tangent of the angle between the base and any face of the pyramid, and also of half the angle between any two side faces.
20. A regular pyramid has for its base a polygon of n sides, and each slant face consists of an isosceles triangle of vertical angle $2a$. If the slant faces are each inclined at an angle β to the base, and at an angle 2γ to one another, shew that

$$\cos\beta = \tan a \cot\frac{\pi}{n} \quad \text{and} \quad \sin\gamma = \sec a \cos\frac{\pi}{n}.$$

CHAPTER XVII

TRIGONOMETRICAL RATIOS OF SMALL ANGLES

AREA OF A CIRCLE. DIP OF THE HORIZON

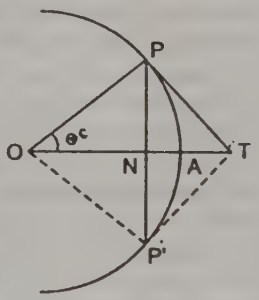
227. *If θ be the number of radians in any angle, which is less than a right angle, then* $\sin\theta, \theta,$ *and* $\tan\theta$ *are in ascending order of magnitude.*

Let TOP be any angle which is less than a right angle.

With centre O and any radius OP, describe an arc PAP' meeting OT in A.

Draw PN perpendicular to OA, and produce it to meet the arc of the circle in P'.

Draw the tangent PT at P to meet OA in T, and join TP'.



The triangles PON and $P'ON$ are equal in all respects, so that $PN = NP'$ and arc PA = arc AP'.

Also the triangles TOP and TOP' are equal in all respects, so that

$$TP = TP'.$$

The straight line PP' is less than the arc PAP', so that NP is $<$ arc PA.

We shall assume that the arc PAP' is less than the sum of PT and TP', so that arc $PA < PT$.

Hence NP, the arc AP, and PT are in ascending order of magnitude.

Therefore $\dfrac{NP}{OP}$, $\dfrac{\text{arc}\,AP}{OP}$, and $\dfrac{PT}{OP}$ are in ascending order of magnitude.

But $\dfrac{NP}{OP} = \sin AOP = \sin\theta,$

$\dfrac{\text{arc}\,AP}{OP}$ = number of radians in $\angle AOP = \theta$ (Art. 21),

and $\dfrac{PT}{OP} = \tan POT = \tan AOP = \tan\theta.$

Hence $\sin\theta, \theta,$ and $\tan\theta$ are in ascending order of magnitude, provided that

$$\theta < \frac{\pi}{2}.$$

The assumption made in the previous article may be proved as follows :

Divide the arc PP' into a large number, n, of equal parts, and let the points of division be $A_1, A_2, A_3, \ldots A_{n-1}$.

Let PA_1 meet TP' in B_1, let A_1A_2 meet it in B_2, A_2A_3 in B_3,... and finally let $A_{n-2}A_{n-1}$ meet it in B_{n-1}.

Since two sides of a triangle are always greater than the third, we have

$$PT + TB_1 > PB_1, \textit{i.e.} > PA_1 + A_1B_1,$$
$$A_1B_1 + B_1B_2 > A_1B_2, \textit{i.e.} > A_1A_2 + A_2B_2,$$
$$A_2B_2 + B_2B_3 > A_2A_3 + A_3B_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_{n-2}B_{n-2} + B_{n-2}B_{n-1} > A_{n-2}A_{n-1} + A_{n-1}B_{n-1},$$

and
$$A_{n-1}B_{n-1} + B_{n-1}P' > A_{n-1}P'.$$

Hence, on adding and cancelling like quantities, we have

$$PT + TB_1 + B_1B_2 + B_2B_3 + \ldots + B_{n-1}P' > PA_1 + A_1A_2 + A_2A_3 + \ldots + A_{n-1}P',$$

i.e.
$$PT + TP' > PA_1 + A_1A_2 + A_2A_3 + \ldots + A_{n-1}P' \ldots (1)$$

Now let the number, n, of subdivisions be indefinitely increased, the right-hand side of (1) becomes ultimately equal to the arc PP', and we thus have

$$PT + TP' > \text{the arc } PP'.$$

228. *Since* $\sin\theta < \theta < \tan\theta$, *we have, by dividing each by the positive quantity* $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Hence $\dfrac{\theta}{\sin\theta}$ always lies between 1 and $\dfrac{1}{\cos\theta}$.

This holds however small θ may be.

Now, when θ is very small, $\cos\theta$ is very nearly unity, and the smaller θ becomes, the more nearly does $\cos\theta$ become unity, and hence the more nearly does $\dfrac{1}{\cos\theta}$ become unity.

Hence, when θ is very small, the quantity $\dfrac{\theta}{\sin\theta}$ lies between 1 and a quantity which differs from unity by an indefinitely small quantity.

In other words, when θ is made indefinitely small the quantity $\dfrac{\theta}{\sin\theta}$, and therfore $\dfrac{\sin\theta}{\theta}$, is ultimately equal to unity, i.e. the smaller an angle becomes the more nearly is its sine equal to the number of radians in it.

This is often shortly expressed thus :

$\sin\theta = \theta$, when θ is very small.

So also $\tan\theta = \theta$, when θ is very small.

Cor. Putting $\theta = \frac{\alpha}{n}$, it follows that, when θ is indefinitely small, n is indefinitely great.

Hence $\dfrac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}$ is unity, when n is indefinitely great.

So $n \sin\frac{\alpha}{n} = \alpha$, when n is indefinitely great.

Similarly, $n \tan\frac{\alpha}{n} = \alpha$, when n is indefinitely great.

229. *In the preceding article it must be particularly noticed that θ is the number of radians in the angle considered.*

The value of $\sin\alpha°$, when α is small, may be found. For, since $\pi^c = 180°$, we have

$$\alpha° = \left(\pi\frac{\alpha}{180}\right)^c.$$

$$\therefore \qquad \sin\alpha° = \sin\left(\frac{\pi\alpha}{180}\right)^c = \frac{\pi\alpha}{180},$$

by the result of the previous article.

230. From the tables it will be seen that the sine of an angle and its circular measure agree to 7 places of decimals so long as the angle is not greater than 18′. They agree to the 5th place of decimals so long as the angle is less than about 2°.

231. If θ be the number of radians in an angle, which is less than a right angle, then $\sin\theta$ is $> \theta - \frac{\theta^3}{4}$ and $\cos\theta$ is $> 1 - \frac{\theta^2}{2}$.

By Art. 227, we have

$$\tan\frac{\theta}{2} > \frac{\theta}{2}.$$

$$\therefore \qquad \sin\frac{\theta}{2} > \frac{\theta}{2}\cos\frac{\theta}{2}.$$

Hence, since $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$,

we have $\sin\theta > \theta\cos^2\frac{\theta}{2}$, *i.e.* $> \theta\left(1 - \sin^2\frac{\theta}{2}\right)$.

But since, by Art. 227,

$$\sin\frac{\theta}{2} < \frac{\theta}{2},$$

therefore $1 - \sin^2\frac{\theta}{2} > 1 - \left(\frac{\theta}{2}\right)^2$, *i.e.* $> 1 - \frac{\theta^2}{4}$.

$\therefore \quad \sin\theta > \theta\left(1 - \frac{\theta^2}{4}\right)$, *i.e.* $> \theta - \frac{\theta^3}{4}$.

Again, $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$;

therefore, since $\sin^2\frac{\theta}{2} < \left(\frac{\theta}{2}\right)^2$,

we have $1 - 2\sin^2\frac{\theta}{2} > 1 - 2\left(\frac{\theta}{2}\right)^2$, *i.e.* $\cos\theta > 1 - \frac{\theta^2}{2}$.

From the results of Part II, Arts. 32 and 33, it will follow that

$$\sin\theta > \theta - \frac{\theta^3}{6}, \text{ and } \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

232. Ex. 1. *Find the values of* $\sin 10'$ *and* $\cos 10'$.

Since $10' = \frac{1^\circ}{6} = \frac{\pi^c}{180 \times 6}$,

we have $\sin 10' = \sin\left(\frac{\pi}{180 \times 6}\right)^c = \frac{\pi}{180 \times 6}$

$$= \frac{3{\cdot}14159265\ldots}{180 \times 6} = {\cdot}0029089 \text{ nearly.}$$

Also $\cos 10' = \sqrt{1 - \sin^2 10'}$

$$= [1 - {\cdot}000008468\ldots]^{1/2}$$

$$= 1 - \frac{1}{2}[{\cdot}000008468\ldots],$$

approximately by the Binomial Theorem,

$$= 1 - {\cdot}000004234\ldots$$

$$= {\cdot}9999958\ldots$$

Ex. 2. *Solve approximately the equation*

$$\sin\theta = {\cdot}52.$$

Since $\sin\theta$ is very nearly equal to $\frac{1}{2}$, θ must be nearly equal to $\frac{\pi}{6}$.

Let then $\theta = \frac{\pi}{6} + x$, where x is small.

$\therefore \quad {\cdot}52 = \sin\left(\frac{\pi}{6} + x\right) = \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x.$

$$= \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x.$$

Since x is very small, we have
$\cos x = 1$ and $\sin x = x$ nearly

$$\therefore \quad \cdot 52 = \frac{1}{2} + \frac{\sqrt{3}}{2}x.$$

$$\therefore \quad x = \cdot 02 \times \frac{2}{\sqrt{3}} \text{ radians} = \frac{\sqrt{3}^c}{75} = 1 \cdot 32^\circ \text{ nearly.}$$

Hence $\theta = 31^\circ\ 19'$ nearly.

EXAMPLES. XL.

$$\left[\pi = 3 \cdot 14159265; \frac{1}{\pi} = \cdot 31831\ldots\right]$$

Find, to 5 places of decimals, the value of

1. $\sin 7'$.
2. $\sin 15''$.
3. $\sin 1'$.
4. $\cos 15'$.
5. $\operatorname{cosec} 8''$.
6. $\sec 5'$.

Solve approximately the equations

7. $\sin\theta = \cdot 01$
8. $\sin\theta = \cdot 48$.
9. $\cos\left(\frac{\pi}{3} + \theta\right) = \cdot 49$.
10. $\cos\theta = \cdot 999$.

11. Find approximately the distance at which a coin, which is 1 cm in diameter, must be placed so as to just hide the moon, the angular diameter of the moon, that is the angle its diameter subtends at the observer's eye, being taken to be $30'$.
12. A person walks in a straight line toward a very distant object, and observes that at three points A, B, and C the angles of elevation of the top of the object are $a, 2a$, and $3a$ respectively; prove that
$$AB = 3BC \text{ nearly.}$$
13. If θ be the number of radians in an angle which is less than a right angle, prove that
$$\cos\theta \text{ is } < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}.$$
14. Prove the theorem of Euler, *viz.* that
$$\sin\theta = \theta \cdot \cos\frac{\theta}{2} \cdot \cos\frac{\theta}{2^2} \cdot \cos\frac{\theta}{2^3} \ldots\ldots \text{and inf.}$$

$$\left[\text{We have } \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2^2 \sin\frac{\theta}{2^2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2}\right]$$
$$= 2^3 \sin\frac{\theta}{2^3}\cos\frac{\theta}{2^3}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2} = \ldots\ldots$$
$$= 2^n \sin\frac{\theta}{2^n} \times \cos\frac{\theta}{2} \cdot \cos\frac{\theta}{2^2} \cdot \cos\frac{\theta}{2^3} \ldots\ldots \cos\frac{\theta}{2^n}.$$

Make n indefinitely great so that, by Art. 228 Cor.,

$$2^n \sin\frac{\theta}{2^n} = \theta.$$

Hence $\sin\theta = \theta \cdot \cos\frac{\theta}{2} \cdot \cos\frac{\theta}{2^2} \cdot \cos\frac{\theta}{2^3} \ldots$ ad inf.]

15. Prove that

$$\left(1 - \tan^2\frac{\theta}{2}\right)\left(1 - \tan^2\frac{\theta}{2^2}\right)\left(1 - \tan^2\frac{\theta}{2^3}\right) \ldots\ldots \text{ad inf.}$$

$$= \theta \cdot \cot\theta.$$

233. Area of a circle.

By Art. 225, the area of a regular polygon of n sides, which is inscribed in a circle of radius R, is

$$\frac{n}{2}R^2 \sin\frac{2\pi}{n}.$$

Let now the number of sides of this polygon be indefinitely increased, the polygon always remaining regular.

It is clear that the perimeter of the polygon must more and more approximate to the circumference of the circle.

Hence, when the number of sides of the polygon is infinitely great, the area of the circle must be the same as that of the polygon.

Now $\frac{n}{2}R^2 \sin\frac{2\pi}{n} = \frac{n}{2}R^2 \cdot \frac{2\pi}{n} \cdot \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} = \pi R^2 \cdot \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}$

$$= \pi R^2 \cdot \frac{\sin\theta}{\theta}, \text{ where } \theta = \frac{2\pi}{n}.$$

When n is made infinitely great, the value of θ becomes infinitely small, and then, by Art. 228, $\frac{\sin\theta}{\theta}$ is unity.

The area of the circle therefore $= \pi R^2 = \pi$ times the square of its radius.

234. *Area of the sector of a circle.*

Let O be the centre of a circle, AB the bounding arc of the sector, and let $\angle AOB = \alpha$ radians.

By Geometry, since sectors are to one another as the arcs on which they stand, we have

$$\frac{\text{area of sector } AOB}{\text{area of whole circle}} = \frac{\text{arc } AB}{\text{circumference}}$$

$$= \frac{R\alpha}{2\pi R} = \frac{\alpha}{2\pi}.$$

$\therefore$ area of sector $AOB = \frac{\alpha}{2\pi} \times$ area of whole circle

$$= \frac{\alpha}{2\pi} \times \pi R^2 = \frac{1}{2}R^2 \cdot \alpha.$$

EXAMPLES. XLI.

[*Assume that* $\pi = 3{\cdot}14159\ldots$, $\frac{1}{\pi} = {\cdot}31831$ and $\log \pi = {\cdot}49715$].

1. Find the area of a circle whose circumference is 74 metres.
2. The diameter of a circle is 10 cm; find the area of a sector whose arc is $22\frac{1}{2}°$.
3. The area of a certain sector of a circle is 10 square cm if the radius of the circle be 3 cm, find the angle of the sector.
4. The perimeter of a certain sector of a circle is 10 cm; if the radius of the circle be 3 cm, find the area of the sector.
5. A strip of paper, two km long and ·003 of *a* cm thick, is rolled up into a solid cylinder; find approximately the radius of the circular ends of the cylinder.
6. A strip of paper, one km long, is rolled tightly up into a solid cylinder, the diameter of whose circular ends is 10 cm; find the thickness of the paper.
7. Give two concentric circles of radii r and $2r$, two parallel tangents to the inner circle cut off an arc from the outer circle; find its length.
8. The circumference of a semicircle is divided into two arcs such that the chord of one is double that of the other. Prove that the sum of the areas of the two segments cut off by these chords is to the area of the semicircle as 27 is to 55.

$$\left[\pi = \frac{22}{7}.\right]$$

9. If each of three circles, of radius a, touch the other two, prove that the area included between them is nearly equal to $\frac{4}{25}a^2$.
10. Six equal circles, each of radius a, are placed so that each touches two others, their centres being all on the circumference of another circle; prove that the area which they enclose is

$$2a^2(3\sqrt{3} - \pi).$$

11. From the vertex A of a triangle a straight line AD is drawn making an angle θ with the base and meeting it at D. Prove that the area common to the circumscribing circles of the triangles ABD and ACD is

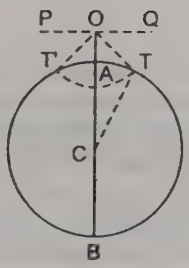
$$\frac{1}{4}(b^2\gamma + c^2\beta - bc\sin A)\operatorname{cosec}^2\theta,$$

where β and γ are the number of radians in the angles B and C respectively.

235. Dip of the Horizon.

Let O be a point at a distance h above the earth's surface. Draw tangents, such as OT and OT', to the surface of the earth. The ends of all these tangents all clearly lie on a circle. This circle is called the **Offing or Visible Horizon.** The angle that each of these tangents OT makes with a horizontal plane POQ is called the **Dip** of the Horizon.

Let r be the radius of the earth, and let B be the other end of the diameter through A.



We then have

$$OT^2 = OC^2 - CT^2$$
$$= (r + h)^2 - r^2$$
$$= h\,(2r + h),$$

so that $OT = \sqrt{h\,(2r + h)}$.

This gives an accurate value for OT.

In all practical cases, however, h is very small compared with r.

[$r = 4000$ miles nearly, and h is never greater, and generally is very considerably less, than 5 miles.]

Hence h^2 is very small compared with hr.

As a close approximation, we have then

$$OT = \sqrt{2hr}.$$

The dip $= \angle\, TOQ$

$$= 90° - \angle\, COT = \angle\, OCT.$$

Also, $\tan OCT = \dfrac{OT}{CT} = \dfrac{\sqrt{2hr}}{r} = \sqrt{\dfrac{2h}{r}}$,

so that, very approximately, we have

$$\angle\, OCT = \sqrt{\frac{2h}{r}} \text{ radians}$$
$$= \left(\sqrt{\frac{2h}{r}}\,\frac{180}{\pi}\right)°$$
$$= \left[\frac{180 \times 60 \times 60}{\pi}\sqrt{\frac{2h}{r}}\right]''.$$

236. Ex. *Taking the radius of the earth as 6400 km; find the dip at the top of a lighthouse which is 45 metres above the sea, and the distance of the offing.*

Here $r = 6400$ km, and $h = 45$ metres $= \dfrac{45}{1000}$ km.

Hence h is very small compared with r, so that

$$OT\sqrt{\frac{2 \times 45 \times 6400}{1000}} = \sqrt{9 \times 64} = 24 \text{ Km.}$$

Also the dip $= \sqrt{\dfrac{2h}{r}}$ radians $= \sqrt{\dfrac{48}{6400}}$ radians

$$= \frac{\sqrt{3}}{20} \times \frac{180}{\pi} \text{ degrees} = 4° 57' 43'' \text{ nearly.}$$

EXAMPLES. XLII.

[Unless otherwise stated, the earth's radius may be taken to be 6400 km.]

1. Find in degrees, minutes, and seconds, the dip of the horizon from the top of a mountain 3200 m high, the earth's radius being 6400 km.
2. The lamp of a lighthouse is 50 metres high; how far off can it be seen ?
3. If the radius of the earth be 4000 km, find the height of a balloon when the dip is 1°.

   Find also the dip when the balloon is 2 km high.
4. From the top of the mast of a ship, which is 20 m above the sea, the light of a lighthouse which is known to be 45 m high can just be seen; prove that its distance is 40 km.
5. From the top of a mast, 20 m above the sea, the top of the mast of another ship can just be seen at a distance of 32 km; prove that the heights of the mast are the same.
6. From the top of the mast of a ship which is 44 metres above the sea-level, the light of a lighthouse can just be seen; after sailing for 15 minutes the light can just be seen from the deck which is 11 metres above the sea-level; prove that the rate of sailing of the ship is nearly 47·48 km per hour towards the lighthouse.
7. Prove that, if the height of the place of observation be n metres, the distance that the observer can see is $\sqrt[8]{\frac{n}{2}}$ km nearly.
8. There are 10 million metres in a quadrant of the earth's circumference. Find approximately the distance at which the top of the Eiffel tower should be visible, its height being 300 metres.
9. Three vertical posts are placed at intervals of a km along a straight canal, each rising to the same height above the surface of the water. The visual line joining the tops of the two extreme posts cuts the middle post at a point 8 cm below its top. Find the radius of the earth to the nearest km.

CHAPTER XVIII

INVERSE CIRCULAR FUNCTIONS

237. If $\sin\theta = a$, where a is a known quantity, we know, from Art. 82, that θ is not definitely known. We only know that θ is some one of a definite series of angles.

The symbol "$\sin^{-1} a$" is used to denote the *smallest* angle, whether positive or negative, that has a for its sine.

The symbol "$\sin^{-1} a$" is read in words as "sine minus one a," and must be carefully distinguished from $\frac{1}{\sin a}$ which would be written, if so desired, in the form $(\sin a)^{-1}$.

It will therefore be carefully noted that "$\sin^{-1} a$" is an angle, and denotes the **smallest numerical** angle whose sin is a.

So "$\cos^{-1} a$" means the smallest numerical angle whose cosine is a. Similarly "$\tan^{-1} a$," "$cot^{-1} a$," "$\text{cosec}^{-1} a$," "$\sec^{-1} a$," "$\text{vers}^{-1} a$," and "$\text{covers}^{-1} a$," are defined.

Hence $\sin^{-1} a$ and $\tan^{-1} a$ (and therefore $\text{cosec}^{-1} a$ and $\cot^{-1} a$) always lie between $-90°$ and $+90°$.

But $\cos^{-1} a$ (and therefore $\sec^{-1} a$) always lies between $0°$ and $180°$.

238. The quantities $\sin^{-1} a, \cos^{-1} a, \tan^{-1} a, \ldots$ are called Inverse Circular Functions.

The symbol $\sin^{-1} a$ is often, especially in foreign mathematical books, written as "arc sin a"; similarly $\cos^{-1} a$ is written "arc cos a," and so for the other inverse ratios.

239. When a is positive, $\sin^{-1} a$ clearly lies between $0°$ and $90°$; when a is negative, it lies between $-90°$ and $0°$.

Ex. $\sin^{-1}\frac{1}{2} = 30°$; $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -60°$.

When a is positive, there are two angles, one lying between $0°$ and $90°$ and the other lying between $-90°$ and $0°$, each of which has its cosine equal to a. [For example both $30°$ and $-30°$ have their cosine equal to $\frac{\sqrt{3}}{2}$.] In this case we take the smallest *positive* angle. Hence $\cos^{-1} a$, when a is positive, lies between $0°$ and $90°$.

So $\cos^{-1} a$, when a is negative, lies between 90° and 180°.

Ex. $\cos^{-1}\frac{1}{\sqrt{2}} = 45°$; $\cos^{-1}\left(-\frac{1}{2}\right) = 120°$.

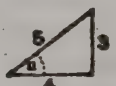
When a is positive, the angle $\tan^{-1} a$ lies between 0° and 90°; when a is negative, it lies between − 90° and 0°.

Ex. $\tan^{-1}\sqrt{3} = 60°$; $\tan^{-1}(-1) = -45°$.

240. Ex. 1. *Prove that* $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$.

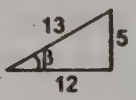
Let $\sin^{-1}\frac{3}{5} = \alpha$, so that $\sin\alpha = \frac{3}{5}$,

and therefore $\cos\alpha = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$.



Let $\cos^{-1}\frac{12}{13} = \beta$, so that $\cos\beta = \frac{12}{13}$

and therefore $\sin\beta = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$.



Let $\sin^{-1}\frac{16}{65} = \gamma$, so that $\sin\gamma = \frac{16}{65}$.

We have then to prove that

$$\alpha - \beta = \gamma,$$

i.e. to shew that $\sin(\alpha - \beta) = \sin\gamma$.

Now $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13} = \frac{36 - 20}{65} = \frac{16}{65} = \sin\gamma.$$

Hence the relation is proved.

Ex. 2. *Prove that* $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$.

Let $\tan^{-1}\frac{1}{3} = \alpha$, so that $\tan\alpha = \frac{1}{3}$,

and let $\tan^{-1}\frac{1}{7} = \beta$, so that $\tan\beta = \frac{1}{7}$.

We have then to shew that

$$2\alpha + \beta = \frac{\pi}{4}.$$

Now $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

$$= \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \frac{6}{8} = \frac{3}{4}.$$

Also, $\tan(2\alpha+\beta)=\dfrac{\tan 2\alpha+\tan\beta}{1-\tan 2\alpha\tan\beta}$

$$=\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}=\frac{21+4}{28-3}=\frac{25}{25}=1=\tan\frac{\pi}{4}.$$

$\therefore \quad 2\alpha+\beta=\dfrac{\pi}{4}$

Ex. 3. *Prove that*

$$4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}.$$

Let $\tan^{-1}\dfrac{1}{5}=\alpha$, so that $\tan\alpha=\dfrac{1}{5}$.

Then $\tan 2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}=\dfrac{\frac{2}{5}}{1-\frac{1}{25}}=\dfrac{5}{12}$,

and $\tan 4\alpha=\dfrac{\frac{10}{12}}{1-\frac{25}{144}}=\dfrac{120}{119}$,

so that $\tan 4\alpha$ is nearly unity, and 4α therefore nearly $\dfrac{\pi}{4}$.

Let $4\alpha=\dfrac{\pi}{4}+\tan^{-1}x$.

$\therefore \quad \dfrac{120}{119}=\tan\left(\dfrac{\pi}{4}+\tan^{-1}x\right)=\dfrac{1+x}{1-x}$ (Art. 100)

$\therefore \quad x=\dfrac{1}{239}$.

Hence $4\tan^{-1}\dfrac{1}{5}-\tan^{-1}\dfrac{1}{239}=\dfrac{\pi}{4}$.

Ex. 4. *Prove that*

$$\tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}.$$

Let $\tan^{-1}a=\alpha$, so that $\tan\alpha=a$,

and let $\tan^{-1}b=\beta$, so that $\tan\beta=b$.

Also, let $\tan^{-1}\left(\dfrac{a+b}{1-ab}\right)=\gamma$, so that $\tan\gamma=\dfrac{a+b}{1-ab}$.

We have then to prove that, $\alpha+\beta=\gamma$.

Now $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{a+b}{1-ab}=\tan\gamma$,

so that the relation is proved.

The above relation is merely the formula

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y},$$

expressed in inverse notation.

For put $\tan x = a$, so that $x = \tan^{-1} a$,

and $\tan y = b$, so that $y = \tan^{-1} b$.

Then $\tan(x+y) = \dfrac{a+b}{1-ab}$.

$$\therefore \quad x + y = \tan^{-1}\frac{a+b}{1-ab},$$

i.e. $$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a+b}{1-ab}.$$

In the above we have tacitly assumed that $ab < 1$, so that $\dfrac{a+b}{1-ab}$ is positive, and therefore $\tan^{-1}\dfrac{a+b}{1-ab}$ lies between 0° and 90°.

If, however, ab be > 1, then $\dfrac{a+b}{1-ab}$ is negative, and therefore according to our definition $\tan^{-1}\dfrac{a+b}{1-ab}$ is a negative angle. Here γ is therefore a negative angle and, since $\tan(\pi+\gamma) = \tan\gamma$, the formula should be

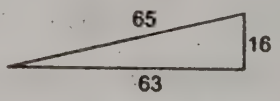
$$\tan^{-1} a + \tan^{-1} b = \pi + \tan^{-1}\frac{a+b}{1-ab}.$$

Ex. 5. *Prove that*

$$\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}.$$

Since $65^2 - 63^2 = 16^2$, we have

$$\cos^{-1}\frac{63}{65} = \tan^{-1}\frac{16}{63}.$$



Also, as in Ex. 1, $\sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{3}{4}$.

We have therefore to shew that

$$\tan^{-1}\frac{16}{63} + 2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{3}{4}.$$

Now $$\tan\left[2\tan^{-1}\frac{1}{5}\right] = \frac{2\tan\left[\tan^{-1}\frac{1}{5}\right]}{1-\tan^2\left[\tan^{-1}\frac{1}{5}\right]} = \frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{5}{12},$$

so that $$2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{5}{12}$$

Thus $\tan\left[\tan^{-1}\frac{16}{63}+2\tan^{-1}\frac{1}{5}\right]=\tan\left[\tan^{-1}\frac{16}{63}+\tan^{-1}\frac{5}{12}\right]$

$$=\frac{\frac{16}{63}+\frac{5}{12}}{1-\frac{16}{63}\cdot\frac{5}{12}}=\frac{192+315}{756-80}=\frac{507}{676}=\frac{3}{4},$$

i.e. $\tan^{-1}\frac{16}{63}+2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{3}{4}.$

Ex. 6. *Solve the equation*

$$\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\tan^{-1}(-7).$$

Taking the tangents of both sides of the equation, we have

$$\frac{\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]+\tan\left[\tan^{-1}\frac{x-1}{x}\right]}{1-\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]\tan\left[\tan^{-1}\frac{x-1}{x}\right]}=\tan\{\tan^{-1}(-7)\}=-7,$$

i.e. $$\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\frac{x-1}{x}}=-7,$$

i.e. $$\frac{2x^2-x+1}{1-x}=-7,$$

so that $x=2$.

This value makes the left-hand side of the given equation positive, so that there is no value of x strictly satisfying the given equation.

The value $x=2$ is a solution of the equation

$$\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\pi+\tan^{-1}(-7).$$

EXAMPLES. XLIII.

[The student should verify the results of some of the following examples (e.g. Nos. 1–4, 8, 9, 12, 13) by an accurate graph.]

Prove that

1. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}.$
2. $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\left(\frac{253}{325}\right).$
3. $\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}.$

4. $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$.

5. $\cos^{-1}x=2\sin^{-1}\sqrt{\frac{1-x}{2}}=2\cos^{-1}\sqrt{\frac{1+x}{2}}$.

6. $2\cos^{-1}\frac{3}{\sqrt{13}}+\cot^{-1}\frac{16}{63}+\frac{1}{2}\cos^{-1}\frac{7}{25}=\pi$.

7. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=45°$.

8. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$.

9. $\tan^{-1}\frac{2}{3}=\frac{1}{2}\tan^{-1}\frac{12}{5}$.

10. $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$.

11. $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

12. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$.

13. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

14. $3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{20}=\frac{\pi}{4}-\tan^{-1}\frac{1}{1985}$.

15. $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$.

16. $\tan^{-1}\frac{120}{119}=2\sin^{-1}\frac{5}{13}$.

17. $\tan^{-1}\frac{m}{n}-\tan^{-1}\frac{m-n}{m+n}=\frac{\pi}{4}$.

18. $\tan^{-1}t+\tan^{-1}\frac{2t}{1-t^2}=\tan^{-1}\frac{3t-t^3}{1-3t^2}$, t being positive,

if $t<\frac{1}{\sqrt{3}}$ or $>\sqrt{3}$, and $=\pi+\tan^{-1}\frac{3t-t^3}{1-3t^2}$ if $t>\frac{1}{\sqrt{3}}$ and $<\sqrt{3}$.

Prove that

19. $\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}=\pi$.

20. $\cot^{-1}\frac{ab+1}{a-b}+\cot^{-1}\frac{bc+1}{b-c}+\cot^{-1}\frac{ca+1}{c-a}=0$.

21. $\tan^{-1}n+\cot^{-1}(n+1)=\tan^{-1}(n^2+n+1)$.

22. $\cos\left(2\tan^{-1}\frac{1}{7}\right) = \sin\left(4\tan^{-1}\frac{1}{3}\right)$.

23. $2\tan^{-1}\left[\tan(45^\circ - \alpha)\tan\frac{\beta}{2}\right] = \cos^{-1}\left[\frac{\sin 2\alpha + \cos\beta}{1 + \sin 2\alpha\cos\beta}\right]$.

24. $\tan^{-1}x = 2\tan^{-1}[\operatorname{cosec}\tan^{-1}x - \tan\cot^{-1}x]$.

25. $2\tan^{-1}\left[\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\right] = \tan^{-1}\frac{\sin\alpha\cos\beta}{\sin\beta + \cos\alpha}$.

26. Shew that

$$\cos^{-1}\sqrt{\frac{a-x}{a-b}} = \sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cot^{-1}\sqrt{\frac{a-x}{x-b}}$$

$$= \frac{1}{2}\sin^{-1}\frac{2\sqrt{(a-x)(x-b)}}{a-b}.$$

27. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = a$, prove that

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos a + \frac{y^2}{b^2} = \sin^2 a.$$

Solve the equations

28. $\tan^{-1}\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} = \beta$.

29. $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$.

30. $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$.

31. $\tan^{-1}(x+1) + \cot^{-1}(x-1) = \sin^{-1}\frac{4}{5} + \cos^{-1}\frac{3}{5}$.

32. $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$.

33. $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$.

34. $\tan^{-1}x + 2\cot^{-1}x = \frac{2}{3}\pi$.

35. $\tan\cos^{-1}x = \sin\cot^{-1}\frac{1}{2}$.

36. $\cot^{-1}x - \cot^{-1}(x+2) = 15^\circ$.

37. $\cos^{-1}\frac{x^2-1}{x^2+1} + \tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}$.

38. $\cot^{-1}x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n-1)$.

39. $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$.

40. $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$.

41. $\tan^{-1}\frac{a}{x} + \tan^{-1}\frac{b}{x} + \tan^{-1}\frac{c}{x} + \tan^{-1}\frac{d}{x} = \frac{\pi}{2}$.

42. $\sec^{-1}\frac{x}{a} - \sec^{-1}\frac{x}{b} = \sec^{-1}b - \sec^{-1}a$.

43. $\operatorname{cosec}^{-1}x = \operatorname{cosec}^{-1}a + \operatorname{cosec}^{-1}b$.

44. $2\tan^{-1}x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}$.

Draw the graphs of

45. $\sin^{-1}x$. [N.B. If $y = \sin^{-1}x$, then $x = \sin y$ and the graph bears the same relation to OY that the curve in Art. 62 bears to OX.]

46. $\cos^{-1}x$. **47.** $\tan^{-1}x$. **48.** $\cot^{-1}x$.

49. $\operatorname{cosec}^{-1}x$. **50.** $\sec^{-1}x$.

51. By obtaining the intersections of the graphs of $\tan x$ and $2x$, shew that the least positive solution of the equation $\tan^{-1}2x = x$ is the circular measure of an angle of approximately 67°.

52. By means of graphs, shew that the solutions of the equations

(1) $\sin x = \frac{1}{100}x$,

(2) $\sin x = \frac{1}{2}x$,

and (3) $\cos x = x^2$

are respectively the circular measures of angles of 178°, 109°, and 47° approximately.

53. By using the tables at the end of this book, shew that nearer approximations to the results of the previous example are the circular measures of angles of 178° 13′, 108° 36′ and 47° 13′ respectively.

CHAPTER XIX

ON SOME SIMPLE TRIGONOMETRICAL SERIES

241. *To find the sum of the sines of a series of angles, the angles being in arithmetical progression.*

Let the angles be

$$\alpha, \alpha + \beta, \alpha + 2\beta, \ldots\ldots \{\alpha + (n-1)\beta\}.$$

Let

$$S \equiv \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin\{\alpha + (n-1)\beta\}.$$

By Art. 97 we have

$$2\sin\alpha \sin\frac{\beta}{2} = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right),$$

$$2\sin(\alpha + \beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right),$$

$$2\sin(\alpha + 2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{5\beta}{2}\right),$$

.....

$$2\sin\{\alpha + (n-2)\beta\}\sin\frac{\beta}{2} = \cos\left\{\alpha + \left(n - \frac{5}{2}\right)\beta\right\} - \cos\left\{\alpha + \left(n - \frac{3}{2}\right)\beta\right\},$$

and

$$2\sin\{\alpha + (n-1)\beta\}\sin\frac{\beta}{2} = \cos\left\{\alpha + \left(n - \frac{3}{2}\right)\beta\right\} - \cos\left\{\alpha + \left(n - \frac{1}{2}\right)\beta\right\}.$$

By adding together these n lines, we have

$$2\sin\frac{\beta}{2} \cdot S = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left\{\alpha + \left(n - \frac{1}{2}\right)\beta\right\},$$

the other terms on the right-hand sides cancelling one another.

Hence, by Art. 94, we have

$$2\sin\frac{\beta}{2}\cdot S = 2\sin\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\sin\frac{n\beta}{2},$$

i.e.

$$S = \frac{\sin\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Ex. By putting $\beta = 2\alpha$, we have

$$\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin(2n-1)\alpha$$
$$= \frac{\sin\{\alpha+(n-1)\alpha\}\sin n\alpha}{\sin\alpha} = \frac{\sin^2 n\alpha}{\sin\alpha}.$$

242. To find the sum of the cosines of a series of angles, the angles being in arithmetical progression.

Let the angles be

$$\alpha, \alpha+\beta, \alpha+2\beta, \ldots \alpha+(n-1)\beta.$$

Let

$$S \equiv \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha+(n-1)\beta\}.$$

By Art. 97, we have

$$2\cos\alpha\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{\beta}{2}\right) - \sin\left(\alpha-\frac{\beta}{2}\right),$$

$$2\cos(\alpha+\beta)\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{3\beta}{2}\right) - \sin\left(\alpha+\frac{\beta}{2}\right),$$

$$2\cos(\alpha+2\beta)\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{5\beta}{2}\right) - \sin\left(\alpha+\frac{3\beta}{2}\right),$$

. . . . . .

$$2\cos\{\alpha+(n-2)\beta\}\sin\frac{\beta}{2} = \sin\left\{\alpha+\left(n-\frac{3}{2}\right)\beta\right\} - \sin\left\{\alpha+\left(n-\frac{5}{2}\right)\beta\right\},$$

and

$$2\cos\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2} = \sin\left\{\alpha+\left(n-\frac{1}{2}\right)\beta\right\} - \sin\left\{\alpha+\left(n-\frac{3}{2}\right)\beta\right\}.$$

By adding together these n lines, we have

$$2S\times\sin\frac{\beta}{2} = \sin\left\{\alpha+\left(n-\frac{1}{2}\right)\beta\right\} - \sin\left\{\alpha-\frac{\beta}{2}\right\},$$

the other terms on the right-hand sides cancelling one another.

Hence, by Art. 94, we have

$$2S \times \sin\frac{\beta}{2} = 2\cos\left\{\alpha + \frac{n-1}{2}\beta\right\}\sin\frac{n\beta}{2},$$

i.e.
$$S = \frac{\cos\left\{\alpha + \frac{n-1}{2}\beta\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

243. Both the expressions for S in Arts. 241 and 242 vanish when $\sin\frac{n\beta}{2}$ is zero, *i.e.* when $\frac{n\beta}{2}$ is equal to any multiple of π,

i.e. when $\frac{n\beta}{2} = p\pi$,

where p is any integer,

i.e. when $\beta = p \cdot \frac{2\pi}{n}$.

Hence the sum of the sines (or cosines) of n angles, which are in arithmetical progression, vanishes when the common difference of the angles is any multiple of $\frac{2\pi}{n}$

Exs. $\cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \cos\left(\alpha + \frac{4\pi}{n}\right) + \ldots$ to n terms $= 0$,

and $\sin\alpha + \sin\left(\alpha + \frac{4\pi}{n}\right) + \sin\left(\alpha + \frac{8\pi}{n}\right) + \ldots$ to n terms $= 0$.

244. Ex. 1. *Find the sum of*

$\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \ldots$ *to n terms.*

We have, by Art. 73,

$$\sin(\alpha+\beta+\pi) = -\sin(\alpha+\beta),$$
$$\sin(\alpha+2\beta+2\pi) = \sin(\alpha+2\beta),$$
$$\sin(\alpha+3\beta+3\pi) = -\sin(\alpha+3\beta),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence the series

$$= \sin\alpha + \sin(\alpha+\beta+\pi) + \sin\{\alpha+2(\beta+\pi)\} + \sin\{\alpha+3(\beta+\pi)\} + \ldots$$

$$= \frac{\sin\left\{\alpha + \frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n(\beta+\pi)}{2}}{\sin\frac{\beta+\pi}{2}}, \text{ by Art. 241,}$$

$$= \frac{\sin\left\{\alpha + \frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n(\beta+\pi)}{2}}{\cos\frac{\beta}{2}}.$$

Ex. 2. Find the sum of the series

$$\cos^3 a + \cos^3 2a + \cos^3 3a + \ldots\ldots \text{ to } n \text{ terms}.$$

By Art. 107, we have

$$\cos 3a = 4\cos^3 a - 3\cos a,$$

so that $4\cos^2 a = 3\cos a + \cos 3a.$

So $4\cos^3 2a = 3\cos 2a + \cos 6a,$

$4\cos^3 3a = 3\cos 3a + \cos 9a,$

.....

Hence, if S be the given series, we have

$$4S = (3\cos a + \cos 3a) + (3\cos 2a - \cos 6a) + (3\cos 3a + \cos 9a) + \ldots$$

$$= 3(\cos a + \cos 2a + \cos 3a + \ldots) + (\cos 3a + \cos 6a + \cos 9a + \ldots)$$

$$= 3\frac{\cos\left\{a + \frac{n-1}{2}\cdot a\right\}\sin\frac{na}{2}}{\sin\frac{a}{2}} + \frac{\cos\left\{3a + \frac{n-1}{2}\cdot 3a\right\}\sin\frac{n\,.\,3a}{2}}{\sin\frac{3a}{2}}$$

$$= 3\frac{\cos\frac{n+1}{2}a\sin\frac{na}{2}}{\sin\frac{a}{2}} + \frac{\cos\frac{3(n+1)}{2}a\sin\frac{3na}{2}}{\sin\frac{3a}{2}}.$$

In a similar manner we can obtain the sum of the cubes of the sines of a series of angles in A.P.

Since $2\sin^2 a = 1 - \cos 2a,$ and $2\cos^2 a = 1 + \cos 2a,$

we can similarly obtain the sum of the squares.

Since again $8\sin^4 a = 2[1 - \cos 2a]^2$

$$= 2 - 4\cos 2a + 2\cos^2 2a = 3 - 4\cos 2a + \cos 4a,$$

we can obtain the sum of the 4th powers of the sines. Similarly for the cosines.

Ex. 3. *Sum to n terms the series*

$$\cos\alpha\sin\beta + \cos 3\alpha\sin 2\beta + \cos 5\alpha\sin 3\beta + \ldots \text{ to } n \text{ terms}.$$

Let S denote the series. Then,

$$2S = \{\sin(\alpha+\beta) - \sin(\alpha-\beta)\} + \{\sin(3\alpha+2\beta) - \sin(3\alpha-2\beta)\} + \{\sin(5\alpha+3\beta) - \sin(5\alpha-3\beta)\} + \ldots$$

$$= \{\sin(\alpha+\beta) + \sin(3\alpha+2\beta) + \sin(5\alpha+3\beta) + \ldots \text{ to } n \text{ terms}\} - \{\sin(\alpha-\beta) + \sin(3\alpha-2\beta) + \sin(5\alpha-3\beta) + \ldots \text{ to } n \text{ terms}\}$$

$$= \frac{\sin\left\{(\alpha+\beta) + \frac{n-1}{2}(2\alpha+\beta)\right\}\sin n\frac{2\alpha+\beta}{2}}{\sin\frac{2\alpha+\beta}{2}} - \frac{\sin\left\{(\alpha-\beta) + \frac{n-1}{2}(2\alpha-\beta)\right\}\sin n\frac{2\alpha-\beta}{2}}{\sin\frac{2\alpha-\beta}{2}},$$ by Art. 241,

$$= \frac{\sin\left\{n\alpha + \frac{n+1}{2}\beta\right\} \sin\frac{n(2\alpha+\beta)}{2}}{\sin\frac{2\alpha+\beta}{2}} - \frac{\sin\left\{n\alpha - \frac{n+1}{2}\beta\right\} \sin\frac{n(2\alpha-\beta)}{2}}{\sin\frac{2\alpha-\beta}{2}}$$

Ex. 4. *$A_1, A_2 \ldots A_n$ is a regular polygon of n sides inscribed in a circle, whose centre is O, and P is any point on the arc $A_n A_1$ such that the angle POA_1 is θ; find the sum of the lengths of the lines joining P to the angular points of the polygon.*

Each of the angles $A_1OA_2, A_2OA_3, \ldots A_nOA_1$ is $\frac{2\pi}{n}$, so that the angles $POA_1, POA_2, \ldots$ are respectively

$$\theta, \theta + \frac{2\pi}{n}, \theta + \frac{4\pi}{n}, \ldots$$

Hence, if r be the radius of the circle, we have

$$PA_1 = 2r \sin\frac{POA_1}{2} = 2r \sin\frac{\theta}{2},$$

$$PA_2 = 2r \sin\frac{POA_2}{2} = 2r \sin\left(\frac{\theta}{2} + \frac{\pi}{n}\right),$$

$$PA_3 = 2r \sin\frac{POA_3}{2} = 2r \sin\left(\frac{\theta}{2} + \frac{2\pi}{n}\right).$$

.....

Hence the required sum

$$= 2r\left[\sin\frac{\theta}{2} + \sin\left(\frac{\theta}{2} + \frac{\pi}{n}\right) + \sin\left(\frac{\theta}{2} + \frac{2\pi}{n}\right) + \ldots\ldots \text{ to } n \text{ terms}\right]$$

$$= 2r \frac{\sin\left[\frac{\theta}{2} + \frac{n-1}{2} \cdot \frac{\pi}{n}\right] \sin\frac{n}{2} \cdot \frac{\pi}{n}}{\sin\frac{\pi}{2n}} \text{ (Art. 241)}$$

$$= 2r \operatorname{cosec}\frac{\pi}{2n} \sin\left[\frac{\pi}{2} + \frac{\theta}{2} - \frac{\pi}{2n}\right]$$

$$= 2r \operatorname{cosec}\frac{\pi}{2n} \cos\left(\frac{\theta}{2} - \frac{\pi}{2n}\right).$$

EXAMPLES. XLIV.

Sum the series :

1. $\cos\theta + \cos 3\theta + \cos 5\theta + \ldots$ to n terms.

2. $\cos\frac{A}{2} + \cos 2A + \cos\frac{7A}{2} + \ldots$ to n terms.

Prove that

3. $$\frac{\sin a + \sin 2a + \sin 3a + \ldots + \sin na}{\cos a + \cos 2a + \ldots + \cos na} = \tan\frac{n+1}{2}a.$$

Prove that

4. $$\frac{\sin a + \sin 3a + \sin 5a + \ldots + \sin(2n-1)a}{\cos a + \cos 3a + \cos 5a + \ldots + \cos(2n-1)a} = \tan na.$$

5. $$\frac{\sin a - \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \text{ to } n \text{ terms}}{\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots \text{ to } n \text{ terms}} = \tan\left\{\alpha + \frac{n-1}{2}(\pi+\beta)\right\}.$$

Sum the following series :

6. $\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + \ldots$ to n terms.

7. $\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \ldots$ to $2n$ terms.

8. $\sin\theta + \sin\frac{n-4}{n-2}\theta + \sin\frac{n-6}{n-2}\theta + \ldots$ to n terms.

9. $\cos x + \sin 3x + \cos 5x + \sin 7x + \ldots + \sin(4n-1)x$.

10. $\sin a \sin 2a + \sin 2a \sin 3a + \sin 3a \sin 4a + \ldots$ to n terms.

11. $\cos a \sin 2a + \sin 2a \cos 3a + \cos 3a \sin 4a + \sin 4a \cos 5a + \ldots$ to $2n$ terms.

12. $\sin a \sin 3a + \sin 2a \sin 4a + \sin 3a \sin 5a + \ldots$ to n terms.

13. $\cos\alpha \cos\beta + \cos 3\alpha \cos 2\beta + \cos 5\alpha \cos 3\beta + \ldots$ to n terms.

14. $\sin^2 a + \sin^2 2a + \sin^2 3a + \ldots$ to n terms.

15. $\sin^2\theta + \sin^2(\theta+a) + \sin^2(\theta+2a) + \ldots$ to n terms.

16. $\sin^3 a + \sin^3 2a + \sin^3 3a + \ldots$ to n terms.

17. $\sin^4 a + \sin^4 2a + \sin^4 3a + \ldots$ to n terms.

18. $\cos^4 a + \cos^4 2a + \cos^4 3a + \ldots$ to n terms.

19. $\cos\theta \cos 2\theta \cos 3\theta + \cos 2\theta \cos 3\theta \cos 4\theta + \ldots$ to n terms.

20. $\sin\alpha \sin(\alpha+\beta) - \sin(\alpha+\beta)\sin(\alpha+2\beta) + \ldots$ to $2n$ terms.

21. From the sum of the series

$$\sin a + \sin 2a + \sin 3a + \ldots \text{ to } n \text{ terms},$$

deduce (by making a very small) the sum of the series

$$1 + 2 + 3 + \ldots + n.$$

22. From the result of the example of Art. 241 deduce the sum of

$$1 + 3 + 5 + \ldots \text{ to } n \text{ terms}.$$

23. If $a = \dfrac{2\pi}{17}$,

prove that $2(\cos a + \cos 2a + \cos 4a + \cos 8a)$
and $2(\cos 3a + \cos 5a + \cos 6a + \cos 7a)$
are the roots of the equation

$$x^2 + x - 4 = 0.$$

24. $ABCD\ldots$ is a regular polygon of n sides which is inscribed in a circle, whose centre is O and whose radius is r, and P is any point on the arc AB such that POA is θ. Prove that

$PA \,.\, PB + PA \,.\, PC + PA \,.\, PD + \ldots + PB \,.\, PC + \ldots$

$$= r^2 \left[2\cos^2\left(\frac{\theta}{2} - \frac{\pi}{2n}\right) \operatorname{cosec}^2 \frac{\pi}{2n} - n \right].$$

25. Two regular polygons, each of n sides, are circumscribed to and inscribed in a given circle. If an angular point of one of them be joined to each of the angular points of the other, then the sum of the squares of the straight lines so drawn is to the sum of the areas of the polygons as

$$2 : \sin\frac{2\pi}{n}.$$

26. $A_1, A_2, \ldots A_{2n+1}$ are the angular points of a regular polygon inscribed in a circle, and O is any point on the circumference between A_1 and A_{2n+1}; prove that

$$OA_1 + OA_3 + \ldots + OA_{2n+1} = OA_2 + OA_4 + \ldots + OA_{2n}.$$

27. If perpendiculars be drawn on the sides of a regular polygon of n sides from any point on the inscribed circle whose radius is a, prove that

$$\frac{2}{n}\Sigma\left(\frac{p}{a}\right)^2 = 3, \text{ and } \frac{2}{n}\Sigma\left(\frac{p}{a}\right)^3 = 5.$$

CHAPTER XX

ELIMINATION

245. It sometimes happens that we have two equations each containing one unknown quantity. In this case there must clearly be a realtion between the constants of the equations in order that the same value of the unknown quantity may satisfy both. For example, suppose we knew that an unknown quantity x satisfied both of the equations

$$ax + b = 0 \text{ and } cx^2 + dx + e = 0.$$

From the first equation, we have

$$x = -\frac{b}{a},$$

and this satisfies the second, if

$$c\left(-\frac{b}{a}\right)^2 + d\left(-\frac{b}{a}\right) + e = 0,$$

i.e. if $b^2c - abd + a^2e = 0.$

This latter equation is the result of eliminating x between the above two equations, and is often called their eliminant.

246. Again, suppose we knew that an angle θ satisfied both of the equations

$$\sin^3\theta = b, \text{ and } \cos^3\theta = c,$$

so that $\sin\theta = b^{1/3}$, and $\cos\theta = c^{1/3}$.

Now we always have, for all values of θ,

$$\sin^2\theta + \cos^2\theta = 1,$$

so that in this case $b^{2/3} + c^{2/3} = 1.$

This is the result of eliminating θ.

247. Between any two equations involving one unknown quantity we can, in theory, always eliminate that quantity. In practice, a considerable amount of artifice and ingenuity is often required in seemingly simple cases.

So, between any three equations involving two unknown quantities, we can theoretically eliminate both of the unknown quantities.

248. Some examples of elimination are appended.

Ex. 1. *Eliminate θ from the equations*

$$a\cos\theta + b\sin\theta = c,$$

and $$d\cos\theta + e\sin\theta = f.$$

Solving for $\cos\theta$ and $\sin\theta$ by cross multiplication, or otherwise, we have

$$\frac{\cos\theta}{bf - ce} = \frac{\sin\theta}{cd - af} = \frac{1}{bd - ae}.$$

$$\therefore \quad 1 = \cos^2\theta + \sin^2\theta = \frac{(bf - ce)^2 + (cd - af)^2}{(bd - ae)^2}.$$

so that $$(bf - ce)^2 + (cd - af)^2 = (bd - ae)^2.$$

Ex. 2. *Eliminate θ between*

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \qquad \ldots(1)$$

and $$\frac{ax\sin\theta}{\cos^2\theta} + \frac{by\cos\theta}{\sin^2\theta} = 0 \qquad \ldots(2)$$

From (2) we have $ax\sin^3\theta = -by\cos^3\theta$.

$$\therefore \quad \frac{\sin\theta}{-(by)^{1/3}} = \frac{\cos\theta}{(ax)^{1/3}} = \frac{\sqrt{\sin^2\theta + \cos^2\theta}}{\sqrt{(by)^{2/3} + (ax)^{2/3}}} = \frac{1}{\sqrt{(by)^{2/3} + (ax)^{2/3}}}$$

Hence $$\frac{1}{\sin\theta} = \frac{\sqrt{(by)^{2/3} + (ax)^{2/3}}}{(by)^{1/3}},$$

and $$\frac{1}{\cos\theta} = \frac{\sqrt{(by)^{2/3} + (ax)^{2/3}}}{(ax)^{1/3}},$$

so that (1) becomes

$$a^2 - b^2 = \sqrt{(by)^{2/3} + (ax)^{2/3}}\left[ax \cdot \frac{1}{(ax)^{1/3}} - by\left\{-\frac{1}{(by)^{1/3}}\right\}\right]$$

$$= \sqrt{(by)^{2/3} + (ax)^{2/3}}\ \{(ax)^{2/3} + (by)^{2/3}\}$$

$$= \{(ax)^{2/3} + (by)^{2/3}\}^{3/2},$$

i.e., $$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

The student who shall afterwards become acquainted with Analytical Geometry will find that the above is the solution of an important problem concerning normals to an ellipse.

Ex. 3. *Eliminate θ from the equations*

$$\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = \cos 2\theta \qquad \ldots(1)$$

and

$$\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 2\sin 2\theta \qquad \ldots(2)$$

Multiplying (1) by $\cos\theta$, (2) by $\sin\theta$, and adding, we have

$$\frac{x}{a} = \cos\theta\cos 2\theta + 2\sin\theta\sin 2\theta$$

$$= \cos\theta + \sin\theta\sin 2\theta = \cos\theta + 2\sin^2\theta\cos\theta \qquad \ldots(3)$$

Multiplying (2) by $\cos\theta$, (1) by $\sin\theta$, and subtracting, we have

$$\frac{y}{b} = 2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta$$

$$= \sin 2\theta\cos\theta + \sin\theta = \sin\theta + 2\sin\theta\cos^2\theta \qquad \ldots(4)$$

Adding (3) and (4), we have

$$\frac{x}{a} + \frac{y}{b} = (\sin\theta + \cos\theta)\,[1 + 2\sin\theta\cos\theta]$$

$$= (\sin\theta + \cos\theta)\,[\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta]$$

$$= (\sin\theta + \cos\theta)^3,$$

so that $\sin\theta + \cos\theta = \left(\frac{x}{a} + \frac{y}{b}\right)^{1/3}$ $\qquad \ldots(5)$

Subtracting (4) from (3), we have

$$\frac{x}{a} - \frac{y}{b} = (\cos\theta - \sin\theta)\,(1 - 2\sin\theta\cos\theta).$$

$$= (\cos\theta - \sin\theta)^3,$$

so that $\cos\theta - \sin\theta = \left(\frac{x}{a} - \frac{y}{b}\right)^{1/3}$ $\qquad \ldots(6)$

Squaring and adding (5) and (6), we have

$$2 = \left(\frac{x}{a} + \frac{y}{b}\right)^{2/3} + \left(\frac{x}{a} - \frac{y}{b}\right)^{2/3}$$

EXAMPLES. XLV.

Eliminate θ from the equations

1. $a\cos\theta + b\sin\theta = c$, and $b\cos\theta - a\sin\theta = d$.
2. $x = a\cos(\theta - \alpha)$, and $y = b\cos(\theta - \beta)$.
3. $a\cos 2\theta = b\sin\theta$, and $c\sin 2\theta = d\cos\theta$.
4. $a\sin a - b\cos a = 2b\sin\theta$, and $a\sin 2a - b\cos 2\theta = a$.
5. $x\sin\theta - y\cos\theta = \sqrt{x^2 + y^2}$, and $\dfrac{\sin^2\theta}{a^2} + \dfrac{\cos^2\theta}{b^2} = \dfrac{1}{x^2 + y^2}$.

6. $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1,$
and $x\sin\theta-y\cos\theta=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}.$

7. $\sin\theta-\cos\theta=p$, and $\operatorname{cosec}\theta-\sin\theta=q$.

8. $x=a\cos\theta+b\cos 2\theta$, and $y=a\sin\theta+b\sin 2\theta$.

9. If $m=\operatorname{cosec}\theta-\sin\theta$, and $n=\sec\theta-\cos\theta$.
prove that $m^{2/3}+n^{2/3}=(mn)^{-2/3}$.

10. Prove that the result of eliminating θ from the equations
$x\cos(\theta+a)+y\sin(\theta+a)=a\sin 2\theta,$
and $y\cos(\theta+a)-x\sin(\theta+a)=2a\cos 2\theta,$
is $(x\cos a+y\sin a)^{2/3}+(x\sin a-y\cos a)^{2/3}=(2a)^{2/3}$

Eliminate θ and ϕ from the equations

11. $\sin\theta+\sin\phi=a$, $\cos\theta+\cos\phi=b$, and $\theta-\phi=a$.

12. $\tan\theta+\tan\phi=x$, $\cot\theta+\cot\phi=y$, and $\theta+\phi=a$.

13. $a\cos^2\theta+b\sin^2\theta=c$, $b\cos^2\phi+a\sin^2\phi=d$,
and $a\tan\theta=b\tan\phi$.

14. $\cos\theta+\cos\phi=a$, $\cot\theta+\cot\phi=b$, and $\operatorname{cosec}\theta+\operatorname{cosec}\phi=?$

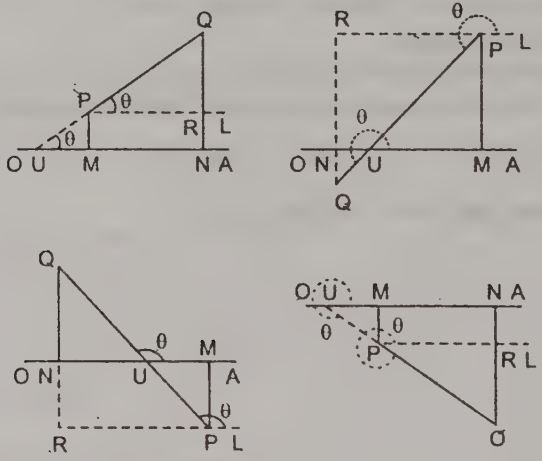
15. $a\sin\theta=b\sin\phi$, $a\cos\theta+b\cos\phi=c$, and $x=y\tan(\theta+\phi)$.

16. $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=1$, $\dfrac{x}{a}\cos\phi+\dfrac{y}{b}\sin\phi=1,$
and $a^2\sin\dfrac{\theta}{2}\sin\dfrac{\phi}{2}+b^2\cos\dfrac{\theta}{2}\cos\dfrac{\phi}{2}=c^2.$

CHAPTER XXI

PROJECTIONS

249. Let PQ be any straight line, and from its ends, P and Q, let perpendiculars be drawn to a fixed straight line OA. Then MN is called the projection of PQ on OA.



If MN be in the same direction as OA, it is positive; if in the opposite direction, it is negative.

250. *If θ be the angle between any straight line PQ and a fixed line OA, the projection of PQ on OA is $PQ\cos\theta$.*

Whatever be the direction of PQ draw, through P, a straight line PL parallel to OA and let it and QN, both produced if necessary, meet in R.

Then, in each figure, the angle LPQ or the angle AUQ is equal to θ.

Also $MN = PR = PQ\cos LPQ = PQ\cos\theta$,

by the definitions of Art. 50.

Similarly, the projection of PQ on a line perpendicular to $OA = RQ$

$= PQ\sin LPQ = PQ\sin\theta$.

The projections of any line PQ on a line to which PQ is inclined at any angle θ, and on a perpendicular line, are therefore PQ $\cos\theta$ and PQ $\sin\theta$.

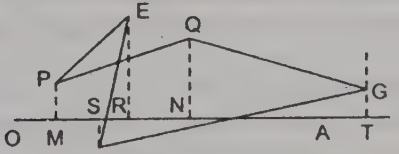
251. We might therefore, in Art. 50, have defined the cosine as the ratio of OP of the projection of OP on the initial line, and, similarly, the sine as the ratio to OP of the projection of OP on a line perpendicular to the initial line.

This method of looking upon the definition of the cosine and sine is often useful.

252. *The projection of PQ upon the fixed line OA is equal to the sum of the projections on OA of* any *broken line beginning at P and ending at Q.*

Let $PEFGQ$ be any broken line joining P and Q.

Draw PM, QN, ER, FS, and GT perpendicular to OA.



The projection of PE is MR and is positive.
The projection of EF is RS and is negative.
The projection of FG is ST and is positive.
The projection of GQ is TN and is negative.
The sum of the projections of the broken line $PEFGQ$ therefore

$$= MR + RS + ST + TN$$
$$= MR - SR + ST - NT$$
$$= MS + SN$$
$$= MN.$$

A similar proof will hold whatever be the positions of P and Q, and however broken the lines joining them may be.

Cor. The sum of the projections of any broken line, joining P to Q, is equal to the sum of the projections of any other broken line joining the same two points; for each sum is equal to the projection of the straight line PQ.

253. *General Proofs, by Projections, of the Addition and Subtraction Theorems.*

Let AOB be the angle A and BOC the angle B. On OC, the bounding line of the anlge $A + B$, take any point P, and draw PN perpendicular to OB and produce it to meet OA in L.

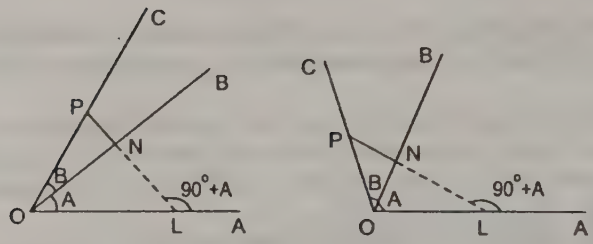
Then $\angle ALP = \angle LNO + \angle AOB = 90° + A$.

(i) *The prove* $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

$OP \cdot \cos(A + B) = OP \cdot \cos AOP$

= projection of OP on OA (Art. 250)

= projection of ON on OA + projection of NP on OA (Art. 252)



$= ON \cos AON + NP \cos ALP$ (Art. 250)

$= OP \cos B \,.\, \cos A + OP \sin B \,.\, \cos (90° + A)$

$= OP\,[\cos A \cos B - \sin A \sin B]$. (Art. 70).

Hence the result (i), on division by OP.

(ii) *To prove* $\sin (A + B) = \sin A \cos B + \cos A \sin B$.

$OP \,.\, \sin (A + B) = OP \,.\, \sin AOP$

= projection of OP on a perpendicular to OA. (Art. 250)

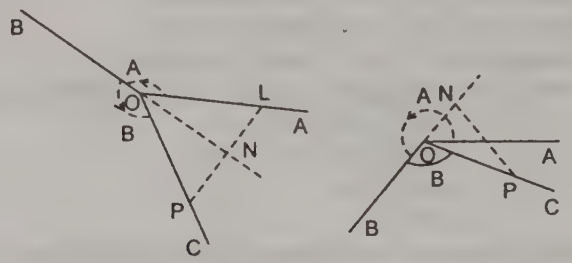
= sum of the projections of ON, NP on the perpendicular to OA (Art. 252)

$= ON \sin A + NP \sin ALP$ (Art. 250)

$= OP \cos B \,.\, \sin A + OP \sin B \,.\, \sin (90° + A)$ (Art. 250)

$= OP\,[\sin A \cos B + \cos A \sin B]$. (Art. 70)

Hence the result (ii).



The above proof holds, as in the subjoined figures, for all positions of the bounding lines OB and OC.

254. In the case of the subtraction theorem, let AOB be the angle A, and let the angle BOC be equal to B described negatively, so that AOC is the angle $A - B$; also OC is inclined to OB at an angle which, with the proper sign prefixed, is $-B$.

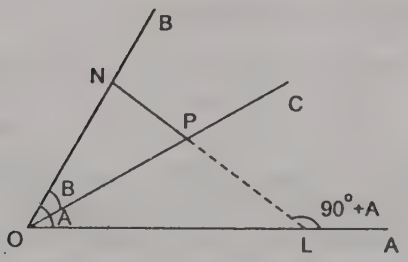
In OC, the bounding line of the angle we are considering, take any point P; draw PN perpendicular to OB and produce it to meet OA in L.

(i) *To prove* $\cos (A - B) = \cos A \cos B + \sin A \sin B$.

$OP \,.\, \cos (A - B) = OP \,.\, \cos AOC$

= projection of OP on OA (Art. 250)

= Projection of ON on OA + projection of NP on OA (Art. 252)



$= ON \cos A + NP \cos (90° + A)$ (Art. 250)

$= OP \cos (-B) \cdot \cos A + OP \sin (-B) \cdot \cos (90° + A)$ (Art. 250)

$= OP \cos B \cos A + OP (-\sin B)(-\sin A)$ (Art. 68, 70)

$= OP [\cos A \cos B + \sin A \sin B]$.

Hence $\cos (A - B) = \cos A \cos B + \sin A \sin B$.

(ii) To prove $\sin (A - B) = \sin A \cos B - \cos A \sin B$.

$OP \cdot \sin (A - B) = OP \cdot \sin AOC$

= projection of OP on a perpendicular to OA (Art. 250)

= sum of the projections of ON, NP on the perpendicular to OA (Art. 252)

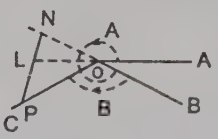
$= ON \sin A + NP \sin (90° + A)$ (Art. 250)

$= OP \cos (-B) \cdot \sin A + OP \sin (-B) \cdot \sin (90° + A)$ (Art. 250)

$= OP \cos B \sin A - OP \sin B \cos A$. (Arts. 68, 70)

$\therefore \sin (A - B) = \sin A \cos B - \cos A \sin B$.

These proofs hold whatever be the positions of the bounding lines OB and OC, as, for example, in the subjoined figure.



MISCELLANEOUS EXAMPLES.

1. Shew that if an angle a be divided into two parts, so that the ratio of the tangents of the parts in λ, the difference x between the parts is given by

$$\sin x = \frac{\lambda - 1}{\lambda + 1} \sin a.$$

2. If $\tan (\pi \cos \theta) = \cot (\pi \sin \theta)$, prove that

$$\cos \left(\theta - \frac{\pi}{4}\right) = \pm \frac{1}{2\sqrt{2}}.$$

3. In any triangle ABC, shew that

$$\frac{a-b}{c}=\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}, \text{ and } \frac{a+b}{c}=\frac{1+\tan\frac{A}{2}\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}.$$

4. An aeroplane is travelling east with a constant velocity of 150 km per hour at a constant height above the ground. At a certain time a man observes it due north of him at an angle of elevation of 9° 30′. At the end of one minute he sees it in a direction 62° east of north. At what height is the aeroplane travelling, and what is the angle of elevation at which the man sees it in the second observed position ?

5. If the sides of a triangle are 51, 35 and 26 feet, find the sides of a triangle, on a base of 41 cm, which shall have the same area and perimeter as the first.

6. Prove that

$$\sin\cot^{-1}\cos\tan^{-1}x=\sqrt{\frac{x^2+1}{x^2+2}}.$$

7. Eliminate θ from the equations

$$\sin(\theta+\alpha)=a,\ \cos^2(\theta+\beta)=b.$$

8. Shew that, whatever be the value of θ, the expression

$$a\sin^2\theta+b\sin\theta\cos\theta+c\cos^2\theta$$

lies between

$$\frac{a+c}{2}+\frac{1}{2}\sqrt{b^2+(a-c)^2}$$

and $$\frac{a+c}{2}-\frac{1}{2}\sqrt{b^2+(a-c)^2}.$$

9. If $\sin x=k\sin(A-x)$,

shew that $\tan\left(x-\frac{A}{2}\right)=\frac{k-1}{k+1}\tan\frac{A}{2}$,

and by means of the tables, solve the equation when $k=3$ and $A=50°$.

10. Express

$$\tan\theta+\tan\left(\theta+\frac{\pi}{3}\right)+\tan\left(\theta+\frac{2\pi}{3}\right)$$

in terms of $\tan 3\theta$.

Hence, or otherwise, solve the equation

$$\tan\theta+\tan\left(\theta+\frac{\pi}{3}\right)+\tan\left(\theta+\frac{2\pi}{3}\right)=3.$$

11. In a triangle ABC, if $\tan\frac{A}{2}, \tan\frac{B}{2}$, and $\tan\frac{C}{2}$ are in arithmetic progression, then $\cos A, \cos B$, and $\cos C$ are also in arithmetic progression.

12. A man standing on the sea shore observes two buoys in the same direction, the line through them making an angle a with the shore. He then walks along the shore a distance a, when he finds the buoys subtend an angle a at his eye; and on walking a further distance b he finds that they again subtend an angle a at his eye. Shew that the distance between the buoys is $\left(a + \frac{b}{2}\right)\sec a - \frac{2a(a+b)}{2a+b}\cos a$, assuming the shore to be straight, and neglecting the height of the man's eye above the sea.

13. The bisectors of the angles of a triangle ABC meet its circumcircle in the points D, E, F respectively. Shew that the area of the triangle DEF is to that of ABC as $R : 2r$.

14. The alternate angles of a regular pentagon are joined forming another regular pentagon; find the ratio of the areas of the two pentagons.

15. If $\phi = \tan^{-1}\frac{x\sqrt{3}}{2k-x}$, and $\theta = \tan^{-1}\frac{2x-k}{k\sqrt{3}}$, prove that one value of $\phi - \theta$ is $30°$.

16. If $m^2 + m'^2 + 2mm'\cos\theta = 1$,

$n^2 + n'^2 + 2nn'\cos\theta = 1$,

and $mn + m'n' + (mn' + m'n)\cos\theta = 0$,

prove that $m^2 + n^2 = \operatorname{cosec}^2\theta$.

17. If x be real, prove that

$$\frac{x^2 - 2x\cos\alpha + 1}{x^2 - 2x\cos\beta + 1} \text{ lies between } \frac{\sin^2\frac{\alpha}{2}}{\sin^2\frac{\beta}{2}} \text{ and } \frac{\cos^2\frac{\alpha}{2}}{\cos^2\frac{\beta}{2}}.$$

18. Prove that the area of a circle exceeds the area of a regular polygon of n sides and of equal perimeter in the ratio of

$$\tan\frac{\pi}{n} : \frac{\pi}{n}.$$

19. If $\frac{\sin(2a-\theta)}{\sin\theta} = 1 + x$, where x is very small, shew that

$$\frac{\cos\theta}{\cos\alpha} = 1 + \frac{1}{2}x\tan^2\alpha, \text{ approximately.}$$

20. If $2\sigma = \alpha + \beta + \gamma + \delta$,

prove that $\cos(\sigma-\alpha)\cos(\sigma-\beta)\cos(\sigma-\gamma)\cos(\sigma-\delta)$

$+ \sin(\sigma-\alpha)\sin(\sigma-\beta)\sin(\sigma-\gamma)\sin(\sigma-\delta)$

$= \cos\alpha\cos\beta\cos\gamma\cos\delta + \sin\alpha\sin\beta\sin\gamma\sin\delta.$

21. Solve completely the equations :

(i) $\tan\alpha \tan(\theta - a) + \tan\beta \tan(\theta - \beta) = \tan\dfrac{\beta - \alpha}{2}(\tan\alpha - \tan\beta)$,

and (ii) $\sin 3\theta = 4 \sin\theta \sin 2\theta \sin 4\theta$.

22. OA is a crank 2 metres long which rotates about O; AB is a connecting rod to B which moves on a straight line passing through O. Find the angles that the crank OA makes with OB when B has described respectively $\frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$ of its total travel from its extreme position, the length of AB being 5 feet.

23. From the top of a cliff, 200 feet high, two ships are observed at sea. The angle of depression of the one is 9° 10′ and it is seen in a direction 30° north of east; the angle of depression of the other is 7° 30′ and it is seen in a direction 25° south of east. What is the distance between the ships, and what is the bearing of the one as seen from the other ?

24. Shew that the radius of the circle, passing through the centre of the inscribed circle of a triangle and any two of the centres of the escribed circles, is equal to the diameter of the circumscribed circle of the triangle.

25. If $\tan\left(\dfrac{\pi}{4} + \dfrac{y}{2}\right) = \tan^3\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$,

prove that $\sin y = \sin x \dfrac{3 + \sin^2 x}{1 + 3\sin^2 x}$.

26. Prove that

$$\sin\beta \sin\gamma \cos^2\alpha \sin(\beta - \gamma) + \text{two similar expressions} = -\sin(\beta - \gamma)\sin(\gamma - \alpha)\sin(\alpha - \beta).$$

27. The legs of a pair of compasses are each 7 cm long, and the pencil leg has a joint at 4 cm from the common end of the two legs. The compasses are used to describe a circle of radius 4 cm and the pencil leg is bent at the joint so that the pencil is perpendicular to the paper. Shew that the angles of inclination of the two legs to the vertical are 19° 5′ and 25° 20′ approximately.

28. A tower stands in a field, whose shape is that of an equilateral triangle and whose side is 80 metres. It subtends angles at the three corners whose tangents are respectively $\sqrt{3} + 1, \sqrt{2}, \sqrt{2}$. Find its height.

29. Two circles of radii r_1 and r_2 cut at an angle a ; shew that the area common to them is

$$(r_1^2 - r_2^2)\tan^{-1}\frac{r_2 \sin a}{r_1 + r_2 \cos a} + r_2^2 a - r_1 r_2 \sin a.$$

30. Find the simplest values of

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, \text{ and } \tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right).$$

31. Eliminate α and β from the equations

$$\sin\alpha+\sin\beta=l,$$
$$\cos\alpha+\cos\beta=m,$$
$$\text{and}\quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2}=n.$$

32. Find, by drawing graphs, how many real roots of the equation $x^2\tan x=1$ lie between 0 and 2π.

33. Shew that

$$\cos 2\alpha=2\sin^2\beta+4\cos(\alpha+\beta)\sin\alpha\sin\beta+\cos 2(\alpha+\beta)$$

34. Prove that $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}=\frac{1}{8}$.

35. Solve the equations

$$\sqrt{3}\sin 2A=\sin 2B,$$
$$\sqrt{3}\sin^2 A+\sin^2 B=\frac{1}{2}(\sqrt{3}-1).$$

36. If the tangents of the angles of a triangle are in arithmetic progression, prove that the squares of the sides are in the ratio

$$x^2(x^2+9):(3+x^2)^2:9(1+x^2),$$

where x is the least or greatest tangent.

37. A, B, C are three points on a horizontal plane in the same straight line, AB being 100 metres and BC 150 metres. The angles of elevation of a balloon observed simultaneously from A, B, C are α, β, γ. Shew that the height h of the balloon in metres is given by

$$h^2(3\cot^2\alpha+2\cot^2\gamma-5\cot^2\beta)=75{,}000$$

38. If p, q, r are the perpendiculars from the vertices of a triangle upon any straight line meeting the sides externally in D, E, F, prove that

$$a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)=4\Delta^2,$$

where Δ is the area of the triangle.

Prove also that $EF=\dfrac{2p\Delta}{(p-q)(p-r)}$.

39. The length of the side of a regular polygon of n sides is $2l$, and the areas of the polygon and of the inscribed and circumscribed circles are A, A_1, and A_2; prove that

$$A_2-A_1=\pi l^2 \text{ and } n^2l^2A_1=\pi A^2.$$

40. Prove that in the triangle whose sides are 31, 56, and 64, one of the angles differs from a right angle by rather less than a minute of angle.

41. Shew that

$$\frac{1+\sin A}{\cos A}+\frac{\cos B}{1-\sin B}=\frac{2\sin A-2\sin B}{\sin(A-B)+\cos A-\cos B}.$$

42. Prove that

$$(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta)\ldots(1+\sec 2^n\theta)=\tan 2^n\theta\,.\cot\theta.$$

43. If the sides of a triangle are in arithmetic progression, and if its greatest angle exceeds the least angle by a, shew that the sides are in the ratio $1-x:1:1+x$, where $x=\sqrt{\dfrac{1-\cos a}{7-\cos a}}$.

44. A tower stands on the edge of a circular lake $ABCD$. The foot of the tower is at D and the angles of elevation of its top at A, B, C are respectively α, β, and γ. If the angles BAC, ACB are each θ, shew that

$$2\cos\theta\cot\beta=\cot\alpha+\cot\gamma.$$

45. The internal bisectors of the angles of a triangle ABC meet the sides in D, E, and F. Shew that the area of the triangle DEF is equal to $2\Delta abc/(b+c)(c+a)(a+b)$.

46. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$,

prove that $x^2+y^2+z^2+2xyz=1$.

47. Eliminate θ from the equations

$$\lambda\cos 2\theta=\cos(\theta+a),$$

and $$\lambda\sin 2\theta=2\sin(\theta+a).$$

48. A circle is described whose diameter is 6 metres; find an equation to determine the angle subtended at the centre by an arc which is such that the sum of the arc and its chord is 8 metres, and solve the equation by a graphic method.

49. Simplify

$$\left\{\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}-\frac{\cos(\alpha+\gamma)}{\cos(\alpha-\gamma)}\right\}^2+\left[\frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)}-\frac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}\right]^2.$$

50. Shew that

$$\sin^2 12^\circ+\sin^2 21^\circ+\sin^2 39^\circ+\sin^2 48^\circ=1+\sin^2 9^\circ+\sin^2 18^\circ.$$

51. If a, b, c are the sides of a triangle, $\lambda a, \lambda b, \lambda c$ the sides of a similar triangle inscribed in the former and θ the angle between the sides a and λa, prove that $2\lambda\cos\theta=1$.

52. The top of a hill is observed from two stations A and B on the same level; A is south of the hill and B is north-east of A. If the angles of elevation from A and B are 9° 30′ and 7° 30′, find the compass bearing of B from the hill.

53. The tangents at B and C to the circumcircle of a triangle ABC meet in A', and O is the circumcentre. If the angle OAA' is θ, prove that

$$2\tan\theta = \cot B - \cot C.$$

54. Find by a geometrical construction the number of values of $\cos(\frac{1}{3}\sin^{-1}a)$. Shew that their product is $-\frac{1}{16}(1-a^2)$.

55. Shew that the expression $\dfrac{\tan(x+a)}{\tan(x-a)}$ cannot lie between the values

$$\tan^2\left(\frac{\pi}{4}-a\right) \text{ and } \tan^2\left(\frac{\pi}{4}+a\right).$$

56. Shew that

$$\cos^4\theta + \cos^4\left(\theta+\frac{2\pi}{n}\right) + \cos^4\left(\theta+\frac{4\pi}{n}\right) + \ldots \text{ to } n \text{ terms} = \frac{3n}{8}.$$

57. If $\{\sin(\alpha-\beta)+\cos(\alpha+2\beta)\sin\beta\}^2 = 4\cos\alpha\sin\beta\sin(\alpha+\beta)$,

prove that $\tan\alpha = \tan\beta\left\{\dfrac{1}{(\sqrt{2}\cos\beta-1)^2}-1\right\}$

α and β being each less than a right angle.

58. Find all the values of x which satisfy the equation

$$\tan(x+\beta)\tan(x+\gamma)+\tan(x+\gamma)\tan(x+\alpha) + \tan(x+\alpha)\tan(x+\beta) = 1.$$

59. ABC is a triangle and D is the foot of the perpendicular from A upon BC. If $BC = 117$ cm, $\angle B = 43°\,14'$, and $\angle C = 61°\,27'$, find the length of AD.

60. The angles of elevation of the top of a mountain from three points A, B and C in a base line are observed to be α, β, γ respectively. Prove that the height of the mountain is

$$(-AB.BC.CA)^{1/2}(BC\cot^2\alpha + CA\cot^2\beta + AB\cot^2\gamma)^{-1/2},$$

where regard is paid to the sense of the lines.

61. If the bisector of the angle C of a triangle ABC cuts AB in D and the circumcircle in E, prove that $CE : DE = (a+b)^2 : c^2$.

62. Eliminate θ from the equations

$$a\tan\theta + b\cot 2\theta = c,$$

and $$a\cot\theta - b\tan 2\theta = c.$$

63. Find, by a graph, an approximate value, correct to half a degree, of the equation $\cot x = \cos 2x$.

64. A man setting out a tennis court uses three strings of lengths 3 m, 4 m, and 4 m and 4·04 m respectively to construct the right angle. Find the errors he makes in the angles of the court.

65. If $n^2 \sin^2(\alpha+\beta) = \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta\cos(\alpha-\beta)$, shew that $\tan\alpha = \dfrac{1 \pm n}{1 \mp n}\tan\beta$.

66. If the expression

$$\frac{A\cos(\theta+\alpha) + B\sin(\theta+\beta)}{A'\sin(\theta+\alpha) + B'\cos(\theta+\beta)}$$

retain the same value for all values of θ, shew that

$$AA' - BB' = (A'B - AB')\sin(\alpha-\beta).$$

67. Shew that the values of θ which are the roots of the equation

$$\sin 2\theta\cos^2(\alpha-\beta) - \sin 2\alpha\cos^2(\beta+\theta) - \sin 2\beta\sin^2(\alpha+\theta) = 0$$

are given by $(2n+1)\dfrac{\pi}{2} - \beta$ and $n\pi + \alpha$, where n is any positive or negative integer.

68. The three medians of a triangle ABC make angles α, β, γ with each other. Prove that

$$\cot\alpha + \cot\beta + \cot\gamma + \cot A + \cot B + \cot C = 0.$$

69. From each of two points, distant $2a$ apart, on one bank of a river the angular elevation of the top of a tower on the opposite bank is a, and from the point midway between these two points the angular elevation of the top of the tower is β. Find in terms of a, α, β the height of the tower and the breadth of the river.

If $a = 100$ m, $\alpha = 22\frac{1}{2}°$, and $\beta = 30°$, shew that the height of the tower is $150\sqrt{2}$ cm.

70. If D, E, F are the points of contact of the inscribed circle with the sides BC, CA, AB of a triangle, shew that if the squares of AD, BE, CF are in arithmetic progression, then the sides of the triangle are in harmonic progression.

71. Shew that half the side of the equilateral triangle inscribed in a circle differs from the side of the regular inscribed heptagon by less than $\dfrac{1}{500}$th of the radius.

72. Shew that the quantity

$$\cos\theta\{\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha}\}$$

always lies between the values $\pm\sqrt{1+\sin^2\alpha}$.

73. Express $8\sin\alpha\sin\beta\sin\gamma\sin\delta$ as a series of eight cosines.

74. If $\sin^2\phi = \dfrac{\cos 2\alpha \cos 2\beta}{\cos^2(\alpha+\beta)}$,

prove that $\tan^2\dfrac{\phi}{2} = \dfrac{\tan\left(\dfrac{\pi}{4} \pm \alpha\right)}{\tan\left(\dfrac{\pi}{4} \pm \beta\right)}$.

75. Given the base a of a triangle, the opposite angle A, and the product k^2 of the other two sides, solve the triangle and shew that there is no such triangle if $a < 2k \sin\dfrac{A}{2}$.

76. At a point O on a horizontal plane the angles of elevation of two points P and Q on the side of a hill are found to be 38° and 25°; the distance of A, the foot of the hill, from O is 500 metres and the distance AQ is 320 metres, the whole figure being in a vertical plane. Prove that the distance PQ is 329 metres approximately, and find the slope of the hill.

77. I_1, I_2, and I_3 are the centres of the circles escribed to ABC, and ρ_1, ρ_2, ρ_3 are the radii of the circles inscribed in the triangles BI_1C, CI_2A, AI_3B. shew that

$$\rho_1 : \rho_2 : \rho_3 :: \sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2}.$$

78. Two circles, the sum of whose radii is a, are placed in the same plane with their centres at a distance $2a$, and an endless string is fully stretched so as partly to surround the circles and to cross between them. Shew that the length of the string is $\left(\dfrac{4\pi}{3} + 2\sqrt{3}\right) a$.

79. Shew that

$$2\tan^{-1}\frac{\sqrt{x^2+a^2}-x+b}{\sqrt{a^2-b^2}} + \tan^{-1}\frac{x\sqrt{a^2-b^2}}{b\sqrt{x^2+a^2+a^2}} + \tan^{-1}\frac{\sqrt{a^2-b^2}}{b}$$

$$= n\pi.$$

80. Given that $-2{\cdot}45$ is an approximate value of x satisfying the equation $3\sin x = 2x + 3$, find a closer approximation.
[Assume that $2{\cdot}45$ radians $= 140°\,22'\,30''$.]

81. Shew that

$$\sin A = \sin(36° + A) - \sin(36° - A) - \sin(72° + A) + \sin(72° - A)$$

82. Find the complete solution of the equations

$$\tan 3\theta + \tan 3\phi = 2,$$

and

$$\tan\phi + \tan\phi = 4.$$

83. If ABC be a triangle, and if

$$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta),$$

then $\cot\theta = \cot A + \cot B + \cot C$.

84. A ship steaming at a speed of 15 km per hour towards a harbour A was observed from a station B, 10 km due west of A, to lie 42° N. of E. If the ship reached the harbour after three-quarters of an hour, find its distance from B when first observed.

85. Shew that the radius of the circle inscribed in the triangle formed by joining the centres of the escribed circles of a triangle ABC is

$$\frac{4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}.$$

86. A polygon of n sides inscribed in a circle is such that its sides subtend angles $2a, 4a, 6a, \ldots 2na$ at the centre; prove that its area is to the area of the regular inscribed polygon of n sides in the ratio

$$\sin na : n \sin a.$$

87. Express the equation

$$\cot^{-1}\left\{\frac{y}{\sqrt{1-x^2-y^2}}\right\} = 2\tan^{-1}\sqrt{\frac{3-4x^2}{4x^2}} - \tan^{-1}\sqrt{\frac{3-4x^2}{x^2}}$$

as a rational integral equation between x and y.

88. If x_1, x_2, x_3, x_4 are the roots of the equation

$$x^4 - x^3\sin 2\beta + x^2\cos 2\beta - x\cos\beta - \sin\beta = 0,$$

prove that

$$\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x_4 = n\pi + \frac{\pi}{2} - \beta,$$

where n is an integer.

89. If $$\frac{\sin(\theta-\beta)\cos\alpha}{\sin(\phi-\alpha)\cos\beta} + \frac{\cos(\alpha+\theta)\sin\beta}{\cos(\phi-\beta)\sin\alpha} = 0,$$

and $$\frac{\tan\theta\tan\alpha}{\tan\phi\tan\beta} + \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = 0,$$

shew that

$$\tan\theta = \frac{1}{2}(\tan\beta + \cot\alpha) \text{ and } \tan\phi = \frac{1}{2}(\tan\alpha - \cot\beta).$$

90. If $\cos 3x = \pm\frac{3}{8}\sqrt{6}$, shew that the three values of $\cos x$ are

$$\frac{1}{2}\sqrt{6}\sin\frac{\pi}{10}, \frac{1}{2}\sqrt{6}\sin\frac{\pi}{6} \text{ and } -\frac{1}{2}\sqrt{6}\sin\frac{3\pi}{10}.$$

91. The base a of a triangle and the ratio $r\,(<1)$ of the sides are given. Shew that the altitude h of the triangle cannot exceed $\frac{ar}{1-r^2}$, and that when h has this value the vertical angle of the triangle is $\frac{\pi}{2} - 2\tan^{-1}r$.

92. A railway-curve, in the shape of a quadrant of a circle, has n telegraph posts at its ends and at equal distances along the curve. A man stationed at a point on one of the extreme radii produced sees the pth and qth posts from the end nearest him in a straight line. Shew that the radius of the curve is $\frac{a}{2}\cos(p+q)\phi \operatorname{cosec} p\phi \operatorname{cosec} q\phi$, where $\phi = \frac{\pi}{4(n-1)}$, and a is the distance from the man to the nearest end of the curve.

93. Shew that the radii of the three escribed circles of a triangle are the roots of the equation

$$x^3 - x^2(4R + r) + xs^2 - rs^2 = 0.$$

94. Eliminate x and y from the equations

$$\cos x + \cos y = a,$$
$$\cos 2x + \cos 2y = b,$$

and $\cos 3x + \cos 3y = c,$

giving the result in a rational form.

95. Sum the series

$\sin\theta \sin 2\theta \sin 3\theta + \sin 2\theta \sin 3\theta \sin 4\theta + \sin 3\theta \sin 4\theta \sin 5\theta + \ldots$ to n terms.

96. In a circle of radius 5 cm the area of a certain segment is 25 square cm. Find graphically the angle that is subtended at the centre by the arc of the segment.

97. Prove that

$$4\sin 27° = (5 + \sqrt{5})^{1/2} - (3 - \sqrt{5})^{1/2}.$$

98. If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) + 1 = 0,$

shew that $\beta - \gamma, \gamma - \alpha$, or $\alpha - \beta$ is a multiple of π.

99. Given the product p of the sines of the angles of a triangle, and the product q of their cosines, shew that the tangents of the angles are the roots of the equation

$$qx^3 - px^2 + (1 + q)x - p = 0.$$

If $p = \frac{1}{8}(3 + \sqrt{3})$ and $q = \frac{1}{8}(\sqrt{3} - 1)$, shew that the angles of the triangle are 45°, 60° and 75°.

100. Observations on the position of a ship are made from a fixed station. At one instant the bearing of the ship is α_1 west of north. Ten minutes later the ship is due north and after a further interval of ten minutes its bearing is α_2 east of north. Assuming that the speed and direction of motion of the ship have not changed, shew that its course is θ east of north where

$$\tan\theta = \frac{2\sin\alpha_1 \sin\alpha_2}{\sin(\alpha_1 - \alpha_2)}.$$

101. A hill on a level plane has the form of a portion of a sphere. At the bottom the surface slopes at an angle a and from a point on the plain distant α from the foot of the hill the elevation of the highest visible point is β. Prove that the height of the hill above the plain is

$$\frac{\alpha \sin\beta \sin^2\frac{\alpha}{2}}{\sin^2\frac{\alpha-\beta}{2}}.$$

102. If D, E, F be the feet of the perpendiculars from ABC on the opposite sides and $\rho, \rho_1, \rho_2, \rho_3$ be the radii of the circles inscribed in the triangles DEF, AEF, BFD, CDE, prove that $r^3\rho = 2R\rho_1\rho_2\rho_3$.

103. O is the centre of a circular field and A any point on its boundary; a horse, tethered by a rope fastened at one end at A, can graze over $\frac{1}{n}$th of the field; if B be the farthest point of the boundary that he can reach and $\angle AOB = \theta$, prove that

$$\sin\theta + (\pi - \theta)\cos\theta = \left(1 - \frac{1}{n}\right)\pi.$$

104. Solve the equation

$$\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\frac{3\sin 2\theta}{5 + 4\cos 2\theta}.$$

105. If $\cos^2\theta = \dfrac{m^2-1}{3}$ and $\tan^3\dfrac{\theta}{2} = \tan a$,

prove that $\cos^{2/3} a + \sin^{2/3} a = \left(\dfrac{2}{m}\right)^{\frac{2}{3}}$

106. A man walks on a horizontal plane a distance a, and then through a distance a at an angle α with his previous direction. After he has done this n times, the change of his direction being always in the same sense, shew that he is distant

$$\frac{a\sin\frac{na}{2}}{\sin\frac{\alpha}{2}}$$

from his starting point, and that this distance makes an angle $(n-1)\dfrac{\alpha}{2}$ with his original direction.

107. Prove that

$$\frac{\tan(\gamma-\delta)}{\tan(\alpha-\beta)}+\frac{\tan(\delta-\beta)}{\tan(\alpha-\gamma)}+\frac{\tan(\beta-\gamma)}{\tan(\alpha-\delta)}$$
$$+\frac{\tan(\gamma-\delta)\tan(\delta-\beta)\tan(\beta-\gamma)}{\tan(\alpha-\beta)\tan(\alpha-\gamma)\tan(\alpha-\delta)}=0$$

108. A meteor moving in a straight line passes vertically above two points, A and B, in a horizontal plane, 1 km. apart. When above A it has altitude 50° as seen from B, and when above B it has altitude 40° as seen from A. Find the distance from A at which it will strike the plane, correct to the nearest metre.

109. The face of a hill is a plane inclined at an angle θ to the horizontal. From two points at the foot of the hill two men walk up it along straight paths lying in vertical planes perpendicular to one another. If they meet after having walked distances a and b respectively, shew that they are then at a vertical height h given by the smaller root of the quadratic.
$(2-\sin^2\theta)h^4-(a^2+b^2)h^2+a^2b^2\sin^2\theta=0.$

110. Shew that, if $\alpha,\beta,\gamma,\delta$ are roots of

$$\tan\left(\theta+\frac{\pi}{4}\right)=3\tan 3\theta,$$

no two of which have equal tangents, then

$$\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=0.$$

111. If $\theta_1,\theta_2,\theta_3,\theta_4$ be roots of the equation

$$\sin(\theta+a)=k\sin 2\theta,$$

no two of which differ by a multiple of 2π, prove that

$$\theta_1+\theta_2+\theta_3+\theta_4=(2n+1)\pi.$$

112. Prove, by means of projections, the theorems of Art. 243.

113. Prove the identities

(i) $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$
$$=4\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}\sin\frac{\alpha+\beta}{2}$$

(ii) $\cos^2\alpha\sin 2(\beta-\gamma)+\cos^2\beta\sin 2(\gamma-\alpha)+\cos^2\gamma\sin 2(\alpha-\beta)$
$$+2\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)=0.$$

114. Shew that the equation

$$\sec\theta+\operatorname{cosec}\theta=c.$$

has two roots between 0 and 2π, if $c^2<8$, and four roots if $c^2>8$.

115. If the external bisectors of the angles of the triangle ABC from a triangle $A_1B_1C_1$, and if the external bisectors of the triangle $A_1B_1C_1$ form a triangle $A_2B_2C_2$, and so on, shew that the angle A_n of the nth derived

triangle is $\frac{\pi}{3} + \left(\frac{-1}{2}\right)^n \left(A - \frac{\pi}{3}\right)$, and that the triangles tend to become equilateral.

116. From a certain station A the angular elevation of a mountain peak P, to the north of A, is α. A hill, of height h above A, is ascended. From B, the top of this hill, the angular elevation of P is β, the bearing of A is δ west of south, and the bearing of P is γ north of A. Shew that the height of P above A is

$$\frac{h \tan\alpha \sin\gamma}{\tan\alpha \sin\gamma - \tan\beta \sin\delta}.$$

117. A man at the bottom of a hill observes an object, 880 metres distant, at the same level as himself. He then walks 200 metres up the hill and observes that the angle of depression of the object is 2° 30′ and that the direction to it makes an angle of 75° with the direction to his starting point. Find to the nearest minute the angle which his path makes with the horizontal.

118. If $2\phi_1, 2\phi_2, 2\phi_3$ are the angles subtended by the circle escribed to the side a of a triangle at the centres of the inscribed circle and the other two escribed circles, prove that

$$\sin\phi_1 \sin\phi_2 \sin\phi_3 = \frac{r_1^2}{16R^2}.$$

119. A regular polygon of n sides is placed with one side in contact with a fixed straight line, and is turned about one extremity of this side until the next side is in contact with the straight line and so on for a complete revolution; shew that the length of the path described by any one of the angular points of the polygon is $\frac{4\pi R}{n} \cot\frac{\pi}{2n}$, where R is the radius of the circle circumscribing the polygon.

Shew also that the sum of the areas of the sectors of the circles described by the angular point is $2\pi R^2$.

120. Eliminate θ and ϕ from the equations

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1,$$

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1,$$

and $$\frac{\cos\theta\cos\phi}{a^2} + \frac{\sin\theta\sin\phi}{b^2} = 0.$$

121. If $S \equiv \sin\theta + 2\sin 2\theta + 3\sin 3\theta + \ldots + n\sin n\theta$,
and $C \equiv \cos\theta + 2\cos 2\theta + 3\cos 3\theta + \ldots + n\cos n\theta$,
prove that

$$4\sin^2\frac{\theta}{2}\cdot S = (n+1)\sin n\theta - n\sin(n+1)\theta,$$

and $$4\sin^2\frac{\theta}{2}\cdot C = -1 + (n+1)\cos n\theta - n\cos(n+1)\theta.$$

122. Prove that

$$\frac{\sin(\theta-\gamma-\alpha)\sin(\theta-\alpha-\beta)}{\sin(\beta-\alpha)\sin(\gamma-\alpha)} + \frac{\sin(\theta-\alpha-\beta)\sin(\theta-\beta-\gamma)}{\sin(\gamma-\beta)\sin(\alpha-\beta)} + \frac{\sin(\theta-\beta-\gamma)\sin(\theta-\gamma-\alpha)}{\sin(\alpha-\gamma)\sin(\beta-\gamma)} = 1.$$

123. If $(\sin^2\phi - \sin^2\psi)\cot\theta + (\sin^2\psi - \sin^2\theta)\cot\phi + (\sin^2\theta - \sin^2\phi)\cot\psi = 0$,
then either the difference of two angles or the sum of all three is a multiple of π.

124. A hill, standing on a horizontal plane, has a circular base and forms part of a sphere. At two points on the plane, distant a and b from the base, the angular elevations of the highest visible points on the hill are θ and ϕ. Prove that the height of the hill is

$$2\left[\frac{\left(b\cot\frac{\phi}{2}\right)^{1/2} - \left(a\cot\frac{\theta}{2}\right)^{1/2}}{\cot\frac{\phi}{2} - \cot\frac{\theta}{2}}\right]^2$$

125. There is a hemispherical dome on the top of a tower; on the top of the dome stands a cross; at a certain point the elevation of the cross is observed to be α, and that of the dome to be β; at a distance a nearer the dome, the cross is seen just above the dome, when its elevation is observed to be γ; prove that the height of the centre of the dome above the ground is

$$\frac{\alpha\sin\gamma}{\sin(\gamma-\alpha)}\cdot\frac{\cos\alpha\sin\beta - \sin\alpha\cos\gamma}{\cos\beta - \cos\gamma}.$$

126. If $\sin^2 A + \sin^2 B + \sin^2 C = 1$, shew that the circumscribed circle of the triangle ABC cuts its nine-point circle orthogonally.

127. A point O is situated on a circle of radius R, and with centre O another circle of radius $\frac{3R}{2}$ is described. Inside the crescent-shaped area intercepted between these circles a circle of radius $\frac{1}{8}R$ is placed. Shew

that if the small circle moves in contact with the original circle of radius R, the length of arc described by its centre in moving from one extreme position to the other is $\frac{7}{12}\pi R$.

128. Eliminate x and y from the equations

$$\sin x + \sin y = a,$$
$$\cos x + \cos y = b,$$

and $$\tan x + \tan y = c.$$

129. If $2\cos n\theta$ be denoted by u_n, shew that

$$u_{n+1} = u_1 u_n - u_{n-1}.$$

Hence shew that

$$2\cos 7\theta = u_1^7 - 7u_1^5 + 14u_1^3 - 7u_1.$$

130. Shew by a graph that $\cdot 74$ is an approximate solution of the equation $\cos x = x$ (where x is measured in radians), and prove that this is the only real root.

Further, by putting $x = \cdot 74 + y$, where y is small, prove that a still nearer value of x is $\cdot 7391$, so that the angle required is $42° 21'$ to the nearest minute.

131. Shew that

$$\frac{\sin(x-\beta)\sin(x-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\sin 2(x-\alpha) + \text{two similar terms} = 0.$$

132. If ABC is a triangle, prove that

$$\sin^3 A\cos(B-C) + \sin^3 B\cos(C-A) + \sin^3 C\cos(A-B)$$
$$= 3\sin A\sin B\sin C.$$

133. A man notices two objects in a straight line due west. After walking a distance c due north he observes that the objects subtend an angle α at his eye; and, after walking a further distance c due north, an angle β. Shew that the distance between the objects is

$$\frac{3c}{2\cot\beta - \cot\alpha}.$$

134. The side of a hill is plane and inclined at an angle α to the horizon; a road on it is in a vertical plane making an angle β with the vertical plane through the line of greatest slope; prove that the inclination of the road to the horizontal is $\tan^{-1}(\tan\alpha\cos\beta)$.

If the road is inclined at an angle γ to the line of greatest slope on the hill, shew that its inclination is $\sin^{-1}(\sin\alpha\cos\gamma)$.

135. Shew that the line joining the incentre to the circumcentre of a triangle ABC is inclined to BC at an angle

$$\tan^{-1}\left(\frac{\cos B + \cos C - 1}{\sin B - \sin C}\right).$$

136. Eliminate θ from the equations

$$x \sin\theta - y\cos\theta = -\sin 4\theta,$$

and

$$x\cos\theta + y\sin\theta = \frac{5}{2} - \frac{3}{2}\cos 4\theta.$$

137. A regular polygon is inscribed in a circle; shew that the arithmetic mean of the squares of the distances of its corners from any point (not necessarily in its plane) is equal to the arithmetic mean of the sum of the squares of the longest and shortest distances of the point from the circle.

138. Three points A, B, C lie in a straight line and AB is to BC as m to n. Through A, B, C are drawn parallel straight lines AX, BY, CZ. A point P moves on AX and a point R on CZ so that at any time t the distance AP is equal to $a_1 + a_2\sin(nt + a)$ and the distance CR is equal to $c_1 + c_2\sin(nt + \gamma)$, and the straight line PR cuts BY in Q. Express the distance BQ in a similar form.

139. Prove that

$$\sin(\beta-\gamma)\sin 3\alpha + \sin(\gamma-\alpha)\sin 3\beta + \sin(\alpha-\beta)\sin 3\gamma$$
$$= 4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\sin(\alpha+\beta+\gamma).$$

140. Prove that

$$\sin(\beta-\gamma)\cos 3\alpha + \sin(\gamma-\alpha)\cos 3\beta + \sin(\alpha-\beta)\cos 3\gamma$$
$$= 4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)\cos(\alpha+\beta+\gamma).$$

141. If $\sin(x+3\alpha)\sin(\beta-\gamma) + \sin(x+3\beta)\sin(\gamma-\alpha)$
$+ \sin(x+3\gamma)\sin(\alpha-\beta) + 4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$,

prove that $x + \alpha + \beta + \gamma = (2n + \frac{1}{2})\pi$.

142. If $A + B + C = 2\pi$, shew that

$$\sin^3 A + \sin^3 B + \sin^3 C = 3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - \sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}.$$

143. A man walks in a horizontal circle round the foot of a flagstaff, which is inclined to the vertical, the foot of the flagstaff being the centre of the circle. The greatest and least angles which the flagstaff subtends at his eye are α and β; and when he is midway between the corresponding positions the angle is θ. If the man's height be neglected, prove that $\tan\theta = \sqrt{\sin^2(\alpha-\beta) + 4\sin^2\alpha\sin^2\beta}/\sin(\alpha+\beta)$.

144. Two lines, inclined at an angle γ, are drawn on an inclined plane and their inclinations to the horizon are found to be α and β respectively; shew that the inclination of the plane to the horizon is

$$\sin^{-1}\{\operatorname{cosec}\gamma\sqrt{\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos\gamma}\}$$

and that the angle between one of the given pair of lines and the line of greatest slope on the inclined plane is

$$\tan^{-1}\left\{\frac{\sin\beta - \sin\alpha\cos\gamma}{\sin\alpha\sin\gamma}\right\}$$

145. Shew that the line joining the orthocentre to the circumcentre of a triangle ABC is inclined to BC at an angle

$$\tan^{-1}\left(\frac{3 - \tan B\tan C}{\tan B - \tan C}\right).$$

146. Eliminate θ from the equations

$$\frac{\cos(\alpha - 3\theta)}{\cos^3\theta} = \frac{\sin(\alpha - 3\theta)}{\sin^3\theta} = m.$$

147. $A_1A_2A_3..A_n$ is a regular polygon of n sides circumscribed to a circle of centre O and radius a. P is any point distant c from O. Shew that the sum of the squares of the perpendiculars from P on the sides of the polygon is $n\left(a^2 + \frac{c^2}{2}\right)$.

148. AB is an arc of a circle which subtends an angle of 2θ at its centre, and the tangents at A and B meet in T. By graphic methods find the value of θ to the nearest degree,

(i) when the area between TA, TB and the arc AB is equal to the area of the circle;

(ii) when the sum of the lengths of TA and TB is equal to the sum of the lengths of the arc AB and the chord AB.

149. Shew that the angles of a triangle satisfy the relations

(i) $\sin^3 A + \sin^3 B + \sin^3 C$

$$= 3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2};$$

(ii) $\sin^4 A + \sin^4 B + \sin^4 C$

$$= \frac{3}{2} + 2\cos A\cos B\cos C + \frac{1}{2}\cos 2A\cos 2B\cos 2C.$$

150. From a point O a man is observed to be walking in a straight path up a hill, and from two sets of observations on his apparent size, made as he passes two points P, Q, it is found that $OP/OQ = \lambda$, and the angle $POQ = \gamma$. The elevations of P and Q above O being α and β, prove that the inclination ϕ of the path to the horizon is given by

$$\sin^2\phi = (\lambda\sin\alpha - \sin\beta)^2/(\lambda^2 - 2\lambda\cos\gamma + 1).$$

151. Eliminate θ from the equations

$$x + a = a(2\cos\theta - \cos 2\theta),$$

and

$$y = a(2\sin\theta - \sin 2\theta).$$

152. If from any point in the plane of a regular polygon perpendiculars are drawn on the sides, shew that the sum of the squares of these perpendiculars is equal to the sum of the squares on the lines joining the feet of the perpendiculars with the centre of the polygon.

153. A horse is tied to a peg in the centre of a rectangular field of sides a and $2a$; if he can graze over just half of the field, shew that the length of the rope by which he is tethered is approximately $\cdot 583a$.

154. Shew that the equation
$$\sin(\theta + \lambda) = a \sin 2\theta + b$$
has four roots whose sum is an odd multiple of two right angles.

155. If θ is a positive acute angle, shew that $\dfrac{\theta}{\sin\theta}$ continually increases, and $\dfrac{\theta}{\tan\theta}$ continually decreases, as θ increases.

156. If $\sin x = m \sin y$ where m is greater than unity, shew that as x increases from zero to a right angle $\dfrac{\tan x}{\tan y}$ continually increases, and that its values, when x is zero and a right angle, are m and ∞ respectively.

157. Prove that
$$\sin^3(\beta-\gamma)\sin^3(\alpha-\delta) + \sin^3(\gamma-\alpha)\sin^3(\beta-\delta) + \sin^3(\alpha-\beta)\sin^3(\gamma-\delta)$$
$$= 3\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\delta)\sin(\beta-\delta)\sin(\gamma-\delta).$$

158. Prove that
$$\Sigma\cos(3\alpha-\beta-\gamma-\delta) = 4\cos(\alpha+\beta-\gamma-\delta)\cos(\alpha+\gamma-\beta-\delta)\cos(\alpha+\delta-\beta-\gamma).$$

159. Shew that
$$\sin(\alpha+\beta+\gamma)\cos\alpha\sin\beta\sin\gamma + \cos(\alpha+\beta+\gamma)\sin\alpha\sin\beta\sin\gamma - \sin(\alpha+\beta+\gamma)\cos\alpha\cos\beta\cos\gamma - \cos(\alpha+\beta+\gamma)\sin\alpha\cos\beta\cos\gamma + \sin(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha) + \cos(\alpha+\beta)\cos(\beta+\gamma)\sin(\gamma+\alpha) = 0.$$

160. Shew that
$$\sin^2\alpha\sin(\beta-\gamma)\sin(\gamma-\delta)\sin(\delta-\beta) - \sin^2\beta\sin(\gamma-\delta)\sin(\delta-\alpha)\sin(\alpha-\gamma) + \sin^2\gamma\sin(\delta-\alpha)\sin(\alpha-\beta)\sin(\beta-\delta) - \sin^2\delta\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha) = 0.$$

161. Simplify the expression $PQ - RS$, where
$$P = x\cos(\alpha+\beta) + y\sin(\alpha+\beta) - \cos(\alpha-\beta),$$
$$Q = x\cos(\gamma+\delta) + y\sin(\gamma+\delta) - \cos(\gamma-\delta),$$
$$R = x\cos(\alpha+\gamma) + y\sin(\alpha+\gamma) - \cos(\alpha-\gamma),$$
and $$S = x\cos(\beta+\delta) + y\sin(\beta+\delta) - \cos(\beta-\delta).$$

162. If $a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\gamma$,
$b^2 + c^2 - 2bc\cos\beta = a^2 + d^2 - 2ad\cos\delta$,
and $ab\sin\alpha + cd\sin\gamma = bc\sin\beta + ad\sin\delta$,
shew that $\cos(\alpha+\gamma) = \cos(\beta+\delta)$.

163. Shew that the solution of the equation

$$\begin{vmatrix} 1, & \cos\theta, & 0, & 0 \cdot \\ \cos\theta, & 1, & \cos\alpha, & \cos\beta \\ 0, & \cos\alpha, & 1, & \cos\gamma \\ 0, & \cos\beta, & \cos\gamma, & 1 \end{vmatrix} = 0$$

is $\theta = n\pi + (-1)^n \sin^{-1}\left\{\dfrac{\sqrt{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\gamma}}{\sin\gamma}\right\}$

164. In any triangle ABC, shew that

$$\cos mA + \cos mB + \cos mC - 1 = \pm 4 \sin\frac{mA}{2} \sin\frac{mB}{2} \sin\frac{mC}{2}.$$

according as m is of the form

$4n + 1$ or $4n + 3$.

165. Shew that, in any triangle ABC,

(i) $a^3 \cos B \cos C + b^3 \cos C \cos A + c^3 \cos A \cos B$
$= abc\,(1 - 2\cos A \cos B \cos C)$,

and

(ii) $\sin 2mA + \sin 2mB + \sin 2mC$
$= (-1)^{m+1} \cdot 4 \sin mA \sin mB \sin mC$.

166. If A, B, C are the angles of a triangle, prove that

$$\tan^{-1}(\cot B \cot C) + \tan^{-1}(\cot C \cot A) + \tan^{-1}(\cot A \cot B)$$
$$= \tan^{-1}\left\{1 + \frac{8\cos A \cos B \cos C}{\sin^2 2A + \sin^2 2B + \sin^2 2C}\right\}.$$

167. Through the angular points A, B, C of a triangle straight lines are drawn making the same angle α with AB, BC, CA respectively; shew that the sides of the triangle thus formed bear to the sides of the triangle ABC the ratio

$$\cos a - \sin a\,(\cot A + \cot B + \cot C) : 1.$$

168. A cylindrical tower is surmounted by a cone; from a point on the ground the angles of elevation of the nearest point of the top of the tower and of the top of the cone are α and β, and from a point nearer to the tower by a distance α these angle are γ and δ. Shew that the heights above the ground of the tops of the tower and cone are
$a \sin\alpha \sin\gamma \operatorname{cosec}(\gamma - \alpha)$ and $a \sin\beta \sin\delta \operatorname{cosec}(\delta - \beta)$,
and that the diameter of the tower is

$$2a \sin\beta \cos\delta \operatorname{cosec}(\delta - \beta) - 2a \sin a \cos\gamma \operatorname{cosec}(\gamma - \alpha).$$

169. In order to find the dip of a stratum of coal below the surface of the ground, vertical borings are made from the angular points A, B, C of a triangle ABC which is in a horizontal plane; the depths of the stratum

at these points are found to be $x, x+y$, and $x+z$ respectively. Shew that the dip, θ, of the stratum, which is assumed to be a plane, is given by

$$\tan\theta \sin A = \sqrt{\frac{y^2}{c^2} + \frac{z^2}{b^2} - \frac{2yz}{bc}\cos A}.$$

170. A tunnel is to be bored from A to B, which are two places on the opposite sides of a mountain. From A and B the elevations of a distant point C are found to be α and β, and the angle ACB is found to be γ; also the lengths AC, BC are known to be a and b. Shew that the height (h) of B above A is $a\sin\alpha - b\sin\beta$, that the length (k) of AB is $\sqrt{a^2+b^2-2ab\cos\gamma}$, and that AB is inclined at $\sin^{-1}\frac{h}{k}$ to the horizontal and at $\sin^{-1}\frac{b\sin\gamma}{k}$ to the line AC.

171. A man walks up a hill of elevation ϕ in a direction making an angle λ with the line of greatest slope; when he has walked up a distance m he observes that α is the angle of depression of an object situated in the horizontal plane through the foot of the hill and in the vertical plane through the path he is taking; after walking a further distance n, he observes that the angle of depression of the same object is β. Shew that the elevation ϕ is given by the equation

$$\left\{\frac{m}{n}(\cot\beta - \cot\alpha) + \cot\beta\right\}^2 + 1 = \operatorname{cosec}^2\phi \sec^2\lambda.$$

172. A, B, C are three mountain peaks of which A is the lowest and B is at a known height h above A. At A the elevations of B and C are found to be β and γ, and the angle between the vertical planes through AB, AC is found to be θ. At B the angle between the vertical planes through BA and BC is found to be ϕ. Shew that the height of C above A is $h\cot\beta\tan\gamma\sin\phi\operatorname{cosec}(\theta+\phi)$.

173. Two straight paths BC, CA on a plane hill-side have lengths a, b respectively and have the same upward gradient of 1 in m (1 vertical in m horizontal) while the gradient from B to A is 1 in p. Shew that the inclination of the plane of the hill to the horizontal is α where

$$4ab\cot^2\alpha = (a+b)^2p^2 - (a-b)^2m^2.$$

174. Shew that the distance between the centres of the inscribed and nine-point circles is equal to $\frac{R}{2} - r$.

Hence deduce Feuerbach's Theorem, that the in-circle and nine-point circles of any triangle touch one another.

175. $ABCD$ is a quadrilateral such that $AB = 3, BC = 4, CD = 5$ and $DA = 6$ cm, and its area is $3\sqrt{3} + \theta$ square cm. Shew that there are two quadrilaterals satisfying these conditions, for which the values of the angle B are respectively 60° and $\cos^{-1}\left[\frac{-1-42\sqrt{3}}{74}\right]$, *i.e.* 60° and 175° 15′ nearly.

176. Eliminate α, β, γ from the equations

$$a\cos\alpha + b\cos\beta + c\cos\gamma = 0,$$
$$a\sin\alpha + b\sin\beta + c\sin\gamma = 0,$$

and

$$a\sec\alpha + b\sec\beta + c\sec\gamma = 0.$$

177. Eliminate θ from the equations

$$\tan(\theta - \alpha) + \tan(\theta - \beta) = x,$$

and

$$\cot(\theta - \alpha) + \cot(\theta - \beta) = y.$$

178. Eliminate ϕ from the equations

$$x\cos 3\phi + y\sin 3\phi = b\cos\phi,$$

and

$$x\sin 3\phi + y\cos 3\phi = b\cos\left(\phi + \frac{\pi}{6}\right).$$

179. Two regular polygons, of m and n sides, are inscribed in the same circle, of radius a; shew that the sum of the squares of all the chords which can be drawn to join a corner of one polygon to a corner of the other is $2mna^2$.

180. There are n stones arranged at equal intervals round the circumference of a circle; compare the labour of carrying them all to the centre with that of heaping them all round one of the stones; and prove that, when the number of stones is indefinitely increased, the ratio is that of $\pi : 4$.

181. By drawing a graph, or otherwise, find the number of roots of the equation

$$x + 2\tan x = \frac{\pi}{2}$$

lying between 0 and 2π, and find the approximate value of the largest of these roots.

Verify your result from the Tables.

182. Find the least positive value of x satisfying the equation

$$\tan x - x = \frac{1}{2}.$$

183. Draw the graph of the function $\sin^2 x$ and shew from it that, if a is small and positive, the equation

$$x - a = \frac{\pi}{2}\sin^2 x$$

has three real roots.

184. Shew that approximations to the larger real roots of the equation

$$ax + b = \tan\frac{\pi cx}{2}$$

are given by $x = \dfrac{m}{c} - \dfrac{2}{\pi(am + bc)}$,

where m is any large odd integer.

185. By a graph determine approximately the numerically smallest positive and negative roots of the equation

$$x^2 \sin \pi x = 1.$$

Prove that the large roots of this equation are given approximately by

$$x = n + \frac{(-1)^n}{n^2\pi},$$

where n is large.

186. Shew that the root of the equation

$$\tan x = 2x,$$

which lies between 0 and $\dfrac{\pi}{2}$, is equal approximately to $1 \cdot 1654$, given that $\tan 1 \cdot 1519 = 2 \cdot 2460$ and $\tan 1 \cdot 1694 = 2 \cdot 3559$.

187. Shew that $\tan\theta$ is always greater than

$$\theta + \frac{\theta^3}{3} + \frac{\theta^5}{15} + \ldots + \frac{\theta^{2n+1}}{4^n - 1} + \ldots,$$

if θ be an acute angle.

188. Shew that the equation

$$\cos(2\theta - \alpha) + a\cos(\theta - \beta) + b = 0,$$

where a, b, α, β are constants, has four sets of roots; and denoting any four roots of different sets by $\theta_1, \theta_2, \theta_3, \theta_4$, prove that

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\alpha$$

is an even multiple of π.

189. The equation

$$\cot(\theta + \alpha) + \cot(\theta + \beta) + \cot(\theta + \gamma)$$
$$= \operatorname{cosec}(\theta + \alpha) + \operatorname{cosec}(\theta + \beta) + \operatorname{cosec}(\theta + \gamma)$$

is satisfied by values of θ equal to θ_1, θ_2, and θ_3, no two of which differ by a multiple of four right angles. Shew that

$$\theta_1 + \theta_2 + \theta_3 + \alpha + \beta + \gamma$$

is equal to a multiple of 2π.

190. Shew that, in general, the equation

$$A \sin^3 x + B \cos^3 x + C = 0$$

has six distinct roots, no two of which differ by 2π, and that the tangent of their semi-sum is $-\frac{A}{B}$.

191. Shew that the equation

$$\tan(\theta - \alpha) + \sec(\theta - \beta) = \cot\gamma$$

has four roots (not differing by multiples of 2π) which satisfy the relation

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2(n\pi + \alpha + \beta - \gamma).$$

192. Shew that if α, β, γ are three values of x satisfying the equation

$$\sin 2\theta (a \sin x + b \cos x) = \sin 2x (a \sin\theta + b \cos\theta)$$

and not differing from one another, or from θ by a multiple of 2π, then

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\theta}{2} + 1 = 0.$$

193. Prove that if $\theta_1, \theta_2, \theta_3, \theta_4$ be four distinct roots of the equation

$$a\cos 2\theta + b \sin 2\theta + c\cos\theta + d = 0,$$

then $\Sigma \sin\dfrac{\theta_2 + \theta_3 + \theta_4 - \theta_1}{2} = 0.$

194. Prove the relation

$$\cos^{-1} x_0 = \frac{\sqrt{1 - x_0^2}}{x_1 . x_2 . x_3 \ldots \text{ad inf.}},$$

where the successive quantities x_r are connected by the relation

$$x_{r+1} = \sqrt{\frac{1}{2}(1 + x_r)}.$$

195. If a, b are positive quantities and if

$$a_1 = \frac{a+b}{2},\ b_1 = \sqrt{a_1 b},\ a_2 = \frac{a_1 + b_1}{2},\ b_2 = \sqrt{a_2 b_1},$$

and so on, shew that

$$a_\infty = b_\infty = \frac{\sqrt{b^2 - a^2}}{\cos^{-1}\dfrac{a}{b}}.$$

Hence shew that the value of π may be found.

196. If the equation

$$a_1 + a_2 \sin x + a_3 \cos x + a_4 \sin 2x + a_5 \cos 2x = 0$$

holds for all values of x, where all the constants $a_1, a_2, \ldots$ are independent of x, then each of these constants must be zero.

197. Prove that if α, β, γ do not differ by a multiple of π, and if

$$\frac{\cos(\alpha+\theta)}{\sin(\beta+\gamma)} = \frac{\cos(\beta+\theta)}{\sin(\gamma+\alpha)},$$

then each fraction is equal to $\dfrac{\cos(\gamma+\theta)}{\sin(\alpha+\beta)}$, and is also equal to ± 1.

198. Prove that $\dfrac{\cot 3x}{\cot x}$ never lies between 3 and $\dfrac{1}{3}$.

199. If A, B, C are the angles of an acute-angled triangle, shew that

$$\tan A \,.\, \tan B \,.\, \tan C > 3\sqrt{3}.$$

200. A long plank, of uniform rectangular section and of breadth b and thickness t, is being lowered with the side b horizontal through a plane circular hole in the deck of a ship. The hole is of diameter d, and the thickness of the deck is h. Prove that the least inclination to the horizontal at which the plank will pass through the deck is $\alpha + \beta$, where

$$\frac{\sin\alpha}{t} = \frac{\sin\beta}{h} = \frac{1}{\sqrt{d^2 + h^2 - b^2}}.$$

201. Two straight lines are taken on the side of a slope, which is an inclined plane, and their inclinations to the horizon are found to be α and β; also the angle between the vertical planes drawn through them is found to be γ. Shew that the inclination of the slope to the horizon is θ, where

$$\sin\gamma \tan\theta = \sqrt{\tan^2\alpha + \tan^2\beta - 2\tan\alpha\tan\beta\cos\gamma},$$

and that the vertical plane through the first of these two lines is inclined to the vertical plane through the line of greatest inclination at an angle

$$\tan^{-1}\left(\frac{\tan\alpha\cos\gamma - \tan\beta}{\tan\alpha\sin\gamma}\right).$$

202. Two roofs are inclined at angles α and β to the horizon respectively; the ridges of the two roofs are horizontal and meet at an angle γ. The intersection of the two roofs is inclined at θ_1 to the horizontal, and the vertical plane through this intersection is inclined at θ_2 to the vertical plane through the ridge of the first roof. Shew that

$$\tan\theta_1 = \frac{\sin\gamma}{\sqrt{\cot^2\alpha + \cot^2\beta + 2\cot\alpha\cot\beta\cos\gamma}},$$

and that $\tan\theta_2 = \dfrac{\cot\alpha\sin\gamma}{\cot\alpha\cos\gamma + \cot\beta}$.

203. A plane is inclined towards the south at an angle α, and on it is a vertical post of height h; when the sun is β west of south and at an elevation γ, shew that the length of the shadow of the post on the inclined plane is

$$\frac{h\cos\gamma}{\sin(\theta+\gamma)}, \text{ where } \tan\theta = \tan\alpha\cos\beta.$$

204. An observer looking up the line of greatest slope of an inclined plane sees a vertical tower due east of him. He walks l metres up the plane in a direction a north of east, and has then reached the level of the foot of the tower and finds its elevation is β. The plane makes an angle γ with the horizontal. Shew that the height of the tower is

$$\frac{l \tan \beta \cos \gamma}{\sqrt{\cot^2 a + \cos^2 \gamma}}.$$

205. The side of a hill slopes towards the south at an angle of 15°; shew that the inclination to the horizon of a road on it which runs in a north-easterly direction is 10° 44′ and that the bearing of a road on it whose slope is 5° to the horizon is 19° 3′ N. of E.

206. A man starts up the line of greatest slope of an inclined plane and after walking 1000 metres finds that he has risen 150 metres. He turns to the right and after going 1000 metres has risen a further 3·3 metres. If the line of greatest slope is due north, find (i) the bearing of the new path, and (ii) the bearing of the original starting-point as seen from the last position.

ANSWERS

Answers have been given to the full number of figures which are likely to emerge from the student's work. They do not imply that in any practical situation the results are reliable to this number of figures.

Even if the data are regarded as exact, answers derived with the use of 5-figure tables have no more than 4-figure acceptability.

I. (Page 3.)

1. $\frac{2}{3}$. **2.** $\frac{301}{360}$. **3.** $\frac{45569}{64800}$.

4. $1\frac{9}{20}$. **5.** $2\frac{3661}{10800}$. **6.** $4\frac{388}{3375}$.

7. $33^g\,33'\,33{\cdot}3''$. **8.** 90^g. **9.** $153^g\,88'\,88{\cdot}8''$.

10. $39^g\,76'\,38{\cdot}8''$. **11.** $261^g\,34'\,44{\cdot}4''$

12. $528^g\,3'\,33{\cdot}3''$. **13.** $1{\cdot}2$ rt. $\angle$; 108°.

14. $\cdot 453524$ rt. $\angle$; 40° 49′ 1·776″.

15. $\cdot 394536$ rt. $\angle$; 35° 30′ 20·664″.

16. $2{\cdot}550809$ rt. $\angle$; 229° 34′ 22·116″.

17. $7{\cdot}590005$ rt. $\angle$; 683° 6′ 1·62″.

28. 5° 33 20″ ; 66° 40′. **29.** $47\frac{7}{19}°$; $42\frac{2}{19}°$.

31. 33° 20′; 10° 48′.

II. (Page 7.)

1. 40212 km. nearly. **2.** 848·2 cm per sec

3. 20·36 km. per hour nearly. **4.** 940600000 km. nearly.

5. 23 km. nearly.

III. (Pages 9, 10.)

1. 60°. **2.** 240°. **3.** 1800°.

4. 57° 17′ 44·8″. **5.** 458° 21′ 58·4″. **6.** 160^g.

7. $233^g\,33'\,33{\cdot}3''$. **8.** 2000^g. **9.** $\frac{\pi}{3}$. **10.** $\frac{221}{360}\pi$

11. $\frac{703}{720}\pi$. **12.** $\frac{3557}{13500}\pi$. **13.** $\frac{79}{36}\pi$.

14. $\frac{3\pi}{10}$. **15.** $\frac{1103}{2000}\pi$. **16.** $1{\cdot}726268\pi$.

17. 81°; 9°. **18.** 24°, 60°, and 96°.

19. 132° 15′ 12·6″. **20.** 30°, 60°, and 90°.

21. $\frac{1}{2}$, $\frac{\pi}{3}$, and $\frac{2\pi}{3} - \frac{1}{2}$ radians.

22. (1) $\frac{3\pi}{5}$; 108°. (2) $\frac{5\pi}{7}$; 128°

(3) $\frac{3\pi}{4}$; 135°. (4) $\frac{5\pi}{6}$; 150°. (5) $\frac{15\pi}{17}$: $158\frac{14°}{17}$

23. 8 and 4. **24.** 10 and 8 **25.** 6 and 8.

26. $\frac{\pi}{3}$. **27.** (1) $\frac{5\pi^c}{12} = 75° = 83\frac{1}{3}g$;

(2) $\frac{7\pi^c}{18} = 70° = 77\frac{7}{9}g$; (3) $\frac{5\pi^c}{8} = 112\frac{1}{2}° = 125^g$.

28. (1) At $7\frac{7}{11}$ and 36 minutes past 4; (2) at $28\frac{4}{11}$ and 48 minutes past 7.

IV. (Pages 12, 13.)

[Take $\pi = 3{\cdot}14159\ldots$ and $\frac{1}{\pi} = {\cdot}31831$.]

1. 20·454° nearly. **2.** $\frac{3}{5}$ radian; 34° 22′ 38·9″.

3. 68·75 cm. nearly. **4.** ·05236 cm. nearly.

5. 24·555 cm. nearly. **6.** 53′ 43″

7. 3959·8 km. nearly. **8.** 10π cm = 31·41159 cm.

9. 5 : 4. **10.** 3·1416.

11. $\frac{4\pi}{35}, \frac{9\pi}{35}, \frac{14\pi}{35}, \frac{19\pi}{35}$, and $\frac{24\pi}{35}$ radians.

12. 65° 24′ 30·4″. **13.** 687·5 m. nearly.

14. 1·5359 m. nearly. **15.** 3151·2 cm. nearly.

16. 32142·9 m. nearly. **17.** 3820 m. nearly.

18. 19·099. **19.** 1788 km. nearly.

20. 386000 km.

VI. (Page 22.)

5. $\frac{\sqrt{15}}{4}, \frac{1}{\sqrt{15}}$, etc. **6.** $\frac{12}{5}; \frac{8}{13}$. **7.** $\frac{11}{60}; \frac{60}{61}; \frac{61}{60}$. **8.** $\frac{3}{5}; \frac{4}{3}$.

9. $\frac{40}{9}; \frac{41}{40}$. **10.** $\frac{3}{5}; \frac{4}{5}; \frac{1}{5}; \frac{5}{3}$. **11.** $\frac{3}{4}$.

12. $\frac{15}{17}; \frac{17}{8}$. **13.** $\frac{1}{2}\sqrt{5}; \frac{3}{5}\sqrt{5}$. **14.** 1 or $\frac{3}{5}$.

15. $\frac{3}{5}$ or $\frac{5}{13}$. **16.** $\frac{5}{13}$. **17.** $\frac{12}{13}$. **18.** $\frac{1}{\sqrt{3}}$ or 1.

19. $\frac{1}{2}$. **20.** $\frac{1}{\sqrt{2}}$. **21.** $1 + \sqrt{2}$.

22. $\frac{2x(x+1)}{2x^2+2x+1}; \frac{2x+1}{2x^2+2x+1}$.

VIII. (Pages 31–33.)

1. 17·32 m; 10 m. **2.** 60 m. **3.** 225 m.
4. 68·3 m. **5.** 146·4 .... m.
6. 367·9 m. **7.** 17·32 m.
8. 115·359 m. **9.** 87·846 ... m.
10. 43·3. . . m.; 75 m. from one of the towers.
11. 94·641. . . m.; 54·641. . . m. **12.** 366. . . metres.
13. 30°. **15.** 13·8564 metres per hour.
16. 23·66. . . m. ; 8·66. . . m.; 28·66. . . m.
17. $32\sqrt{5}$ (= 71·55...) m. **19.** 10 miles per hour.
20. 86·6. . . metres. **21.** 692·8. . . metres.

IX. (Page 64.)

1. $\frac{2250}{6289}\pi, \frac{2500}{6289}\pi$ and $\frac{81}{331}\pi$ radians.

2. 68° 45′ 17·8″. **4.** $\frac{2xy}{x^2+y^2}; \frac{2xy}{x^2-y^2}$.

8. $\frac{1}{\tan^4 A} - \tan^4 A$. **9.** $\theta = 60°$

10. In $1\frac{1}{2}$ minutes.

X. (Pages 52.)

4. − ·366...; 2·3094... **5.** − 1·366... ; − 2·3094.
6. 0; 2. **7.** 1·4142...; −2.
8. 1·366. . . ; − 2·3094... **9.** 45° and 135°.
10. 120° and 240°. **11.** 135° and 315°.
12. 150° and 330°. **13.** 150° and 210°.
14. 210° and 330°. **15.** −cos 25°.
19. sin 6°. **17.** − tan 43°. **18.** sin 12°.
19. sin 17°. **20.** − cot 24°. **21.** cos 33°.
22. − cos 28°. **23.** cot 25°. **24.** cos 30°.
25. cot 26°. **26.** − cosec 23°. **27.** − cosec 36°.
28. negative. **29.** negative. **30.** positive.
31. zero. **32.** positive. **33.** positive
34. positive. **35.** negative.
36. $\frac{1}{\sqrt{3}}$ and $\frac{-\sqrt{2}}{\sqrt{3}}; \frac{-1}{\sqrt{3}}$ and $\frac{\sqrt{2}}{\sqrt{3}}$.

XI. (Pages 57-59)

1. $n\pi + (-1)^n \frac{\pi}{6}$. **2.** $n\pi - (-1)^n \frac{\pi}{3}$.

3. $n\pi + (-1)^n \frac{\pi}{4}$. 4. $2n\pi \pm \frac{2\pi}{3}$.

5. $2n\pi \pm \frac{\pi}{6}$. 6. $2n\pi \pm \frac{3\pi}{4}$. 7. $n\pi + \frac{\pi}{3}$.

8. $n\pi + \frac{3\pi}{4}$. 9. $n\pi + \frac{\pi}{4}$. 10. $2n\pi \pm \frac{\pi}{3}$.

11. $n\pi + (-1)^n \frac{\pi}{3}$. 12. $n\pi \pm \frac{\pi}{2}$. 13. $n\pi \pm \frac{\pi}{3}$.

14. $n\pi \pm \frac{\pi}{6}$. 15. $n\pi \pm \frac{\pi}{3}$. 16. $n\pi \pm \frac{\pi}{4}$.

17. $n\pi \pm \frac{\pi}{6}$. 18. $(2n+1)\pi + \frac{\pi}{4}$. 19. $2n\pi - \frac{\pi}{6}$.

20. 105° and 45°; $\left(n + \frac{m}{2}\right)\pi \pm \frac{\pi}{6} + (-1)^m \frac{\pi}{12}$, and

$$\left(\frac{m}{2} - n\right)\pi \mp \frac{\pi}{6} + (-1)^m \frac{\pi}{12},$$

where m and n are any integers.

21. $187\frac{1}{2}°$ and $142\frac{1}{2}°$;

$$\left(n + \frac{m}{2}\right)\pi + \frac{\pi}{8} \pm \frac{\pi}{12} \text{ and } \left(n - \frac{m}{2}\right)\pi - \frac{\pi}{8} \pm \frac{\pi}{12}.$$

22. (1) 60° and 120°; (2) 120° and 240°; (3) 30° and 210°.

23. (1) 2; (2) 1; (3) 1; (4) 1; (5) 1.

XII. (Page 60.)

1. $n\pi + (-1)^n \frac{\pi}{6}$. 2. $2n\pi \pm \frac{2\pi}{3}$.

3. $n\pi + (-1)^n \frac{\pi}{3}$. 4. $\cos\theta = \frac{\sqrt{5}-1}{2}$.

5. $n\pi + (-1)^n \frac{\pi}{10}$ or $n\pi - (-1)^n \frac{3\pi}{10}$ (Art. 120).

6. $\theta = 2n\pi \pm \frac{\pi}{3}$. 7. $\theta = n\pi + \frac{\pi}{4}$ or $n\pi + \frac{\pi}{3}$.

8. $\theta = n\pi + \frac{2\pi}{3}$ or $n\pi + \frac{5\pi}{6}$. 9. $\tan\theta = \frac{1}{a}$ or $-\frac{1}{b}$.

10. $\theta = n\pi \pm \frac{\pi}{4}$. 11. $\theta = 2n\pi$ or $2n\pi + \frac{\pi}{4}$.

12. $n\pi \pm \frac{\pi}{6}$. 13. $n\pi$ or $2n\pi \pm \frac{\pi}{3}$.

14. $2n\pi \pm \frac{\pi}{3}$ or $2n\pi \pm \frac{\pi}{6}$. 15. $\sin\theta = 1$ or $-\frac{1}{3}$.

16. $\frac{n\pi}{5} + (-1)^n \frac{\pi}{20}$. 17. $\frac{n\pi}{4}$ or $\frac{(2n+1)\pi}{10}$.

18. $2n\pi$ or $\frac{(2n+1)\pi}{5}$. 19. $\frac{2r\pi}{m-n}$ or $\frac{2r\pi}{m+n}$.

20. $\left(2n + \frac{1}{2}\right)\frac{\pi}{5}$ or $2n\pi - \frac{\pi}{2}$. 21. $2n\pi$ or $\frac{2n\pi}{9}$.

22. $\left(2r + \frac{1}{2}\right)\frac{\pi}{m+n}$ or $\left(2r - \frac{1}{2}\right)\frac{\pi}{m-n}$.

23. $\left(n + \frac{1}{2}\right)\frac{\pi}{9}$. 24. $\left(m + \frac{1}{2}\right)\frac{\pi}{n+1}$.

25. $\frac{n\pi}{4} \pm \sqrt{1 + \frac{n^2\pi^2}{16}}$. 26. $n\pi \pm \frac{\pi}{6}$.

27. $\left(n + \frac{1}{2}\right)\frac{\pi}{3} \pm \frac{\alpha}{3}$. 28. $\left(n + \frac{1}{2}\right)\frac{\pi}{4}$.

29. $\frac{n\pi}{3} \pm \frac{\alpha}{3}$. 30. $n\pi \pm \frac{\pi}{6}$. 31. $\left(r + \frac{1}{2}\right)\frac{\pi}{m-n}$.

32. $\tan\theta = \frac{2n + 1 \pm \sqrt{4n^2 + 4n - 15}}{4}$, where $n > 1$ or < -2.

33. $\theta = \left(m + \frac{n}{2}\right)\pi \pm \frac{\pi}{6} + (-1)^n \frac{\pi}{12}$;

$\phi = \left(m - \frac{n}{2}\right)\pi \pm \frac{\pi}{6} - (-1)^n \frac{\pi}{12}$.

34. $\frac{1}{5}\left[(6m - 4n)\pi \pm \frac{\pi}{2} \mp \frac{2\pi}{3}\right]$; $\frac{1}{5}\left[(6n - 4m)\pi \pm \pi \mp \frac{\pi}{3}\right]$.

35. 45° and 60°. 36. $\frac{1}{3}$ or $\frac{5}{3}$.

37. $\pm\frac{1}{3}\sqrt{5}$; $\pm\frac{1}{2}\sqrt{5}$.

XIII. (Pages 64-65)

1. $-\frac{133}{205}$; $-\frac{84}{205}$. 2. $\frac{1596}{3445}$; $\frac{3444}{3445}$.

3. $\frac{220}{221}$; $\frac{171}{221}$; $\frac{220}{21}$.

XIV. (Pages 69.)

30. $2\sin(\theta + n\phi)\sin\frac{3\phi}{2}$. 31. $2\sin(\theta + n\phi)\cos\frac{\phi}{2}$.

XV. (Pages 70.)

1. $\cos 2\theta - \cos 12\theta$. 2. $\sin 12\theta - \sin 2\theta$.
3. $\cos 14\theta + \cos 8\theta$. 4. $\cos 12^\circ - \cos 120^\circ$.

XVI. (Pages 72, 73.)

1. $3; \frac{9}{13}$. 3. 1.

XVII. (Pages 77.)

1. (1) $\pm \frac{24}{25}$; (2) $\pm \frac{120}{169}$; (3) $\frac{2016}{4225}$.
2. (1) $\frac{161}{289}$; (2) $-\frac{7}{25}$; (3) $\frac{119}{169}$. 3. *a*.

XVIII. (Pages 89, 91.)

1. $\frac{\pm 2\sqrt{2} \pm \sqrt{3}}{6}; \frac{\pm 7\sqrt{3} \pm 4\sqrt{2}}{18}$.
2. $\pm \frac{13}{12}; \pm \frac{\sqrt{13}}{2}$ or $\pm \frac{\sqrt{13}}{3}; \frac{169}{120}$.
3. $\frac{16}{305}; \frac{49}{305}$. 4. $\frac{7}{5\sqrt{2}}$. 5. $\pm \frac{1}{3}; \pm \frac{3}{4}$. 6. $\pm \frac{3}{4}$.
7. $\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}; \frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}}; \sqrt{2}-1; -(\sqrt{2}+1)+\sqrt{4+2\sqrt{2}}$
8. $\sqrt{\frac{4-a^2-b^2}{a^2-b^2}}$. 23. + and –. 24. – and –.
25. – and –.
29. (1) $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$; (2) $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$;
(3) $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{\pi}{4}$; (4) $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$.
30. (1) $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{\pi}{4}$; (2) $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$;
(3) $2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$.

XIX. (Pages 93-94.)

12. The sine of the angle is equal to 2 sin 18°.
13. $\frac{n\pi}{8}$ or $\left(2n \pm \frac{1}{3}\right)\frac{\pi}{8}$.

XXI. (Pages 104)

1. $\frac{n\pi}{4}$ or $\frac{1}{3}\left(2n\pi \pm \frac{\pi}{3}\right)$

2. $\left(n + \frac{1}{2}\right)\frac{\pi}{4}$ or $\left(2n \pm \frac{1}{3}\right)\frac{\pi}{3}$.

3. $\left(n + \frac{1}{2}\right)\frac{\pi}{2}$ or $2n\pi$.

4. $\left(n + \frac{1}{2}\right)\frac{\pi}{3}$ or $n\pi + (-1)^n \frac{\pi}{6}$.

5. $\frac{2n\pi}{3}$ or $\left(n + \frac{1}{4}\right)\pi$ or $\left(2n - \frac{1}{2}\right)\pi$.

6. $\frac{n\pi}{3}$ or $\left(2n \pm \frac{1}{3}\right)\frac{\pi}{4}$.

7. $\left(n + \frac{1}{2}\right)\frac{\pi}{2}$ or $2n\pi \pm \frac{2\pi}{3}$.

8. $\frac{n\pi}{3}$ or $\left(n \pm \frac{1}{3}\right)\pi$.

9. $2n\pi$ or $\left(\frac{2n}{3} + \frac{1}{2}\right)\pi$.

10. $n\pi + (-1)^n \frac{\pi}{6}$ or $n\pi + (-1)^n \frac{\pi}{10}$ or $n\pi - (-1)^n \frac{3\pi}{10}$.

11. $\left(n + \frac{1}{2}\right)\frac{\pi}{8}$ or $\left(n + \frac{1}{2}\right)\frac{\pi}{2}$.

12. $n\pi$ or $\frac{1}{n-1}\left[m\pi - (-1)^m \frac{\pi}{6}\right]$

13. $2m\pi$ or $\frac{4m\pi}{n+1}$.

14. $\frac{2r\pi}{m+n}$ or $(2r+1)\frac{\pi}{m-n}$.

15. $(2r+1)\frac{\pi}{m+n}$.

16. $m\pi$ or $\frac{m\pi}{n-1}$ or $\left(m + \frac{1}{2}\right)\frac{\pi}{n}$.

17. $2n\pi - \frac{\pi}{2}; \frac{1}{5}\left(2n\pi - \frac{\pi}{2}\right)$.

18. $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}$.

19. $2n\pi + \frac{\pi}{4}$.

20. $n\pi + \frac{\pi}{6} + (-1)^n \frac{\pi}{4}$.

21. $2n\pi + \frac{\pi}{4} \pm A$.

22. $-21° 48' + n \cdot 180° + (-1)^n [68° 12']$.

23. $2n \cdot 180° + 78° 58'; 2n \cdot 180° + 27° 18'$.

24. $n \cdot 180° + 45°; n \cdot 180° + 26° 34'$

25. $2n\pi + \frac{2\pi}{3}$.

26. $2n\pi$ or $2n\pi + \frac{\pi}{2}$.

27. $2n\pi + \frac{\pi}{2}$ or $2n\pi - \frac{\pi}{3}$.

28. $2n\pi + \frac{\pi}{6}$.

29. $n\pi$.

30. $\sin\theta = \frac{\pm\sqrt{17} - 1}{8}$.

31. $\cos\theta = \frac{\sqrt{17} - 3}{4}$.

32. $n\pi \pm \frac{\pi}{3}$ or $n\pi + \frac{\pi}{2}$. 33. $2n\pi \pm \frac{\pi}{3}$; $2n\pi \pm \frac{\pi}{4}$.

34. $\left(n + \frac{1}{4}\right)\frac{\pi}{2}$. 35. $n\pi \pm \frac{\pi}{4}$. 36. $n\pi + \frac{\pi}{4}$.

37. $\theta = \frac{n\pi}{2}$ or $n\pi \pm \frac{\pi}{3}$; also $\theta = n\pi \pm \frac{\alpha}{2}$, where $\cos\alpha = \frac{1}{3}$.

38. $\left(n + \frac{1}{3}\right)\frac{\pi}{3}$. 39. $n\pi \pm \frac{\pi}{3}$.

XXIII. (Page 116)

In obtaining the answers to Nos. 8–13, and in similar cases in later sets of Examples, seven-figure tables have been used. If the five-figure tables at the end of this book are used, less accurate results will in many cases be obtained.

1. $\bar{1}\cdot 90309$; $\bar{3}\cdot 4771213$; $\bar{2}\cdot 0334239$; $\bar{1}\cdot 4650389$.
2. $\cdot 1553361$; $\bar{2}\cdot 1241781$; $\cdot 5388340$; $\bar{1}\cdot 0759623$.
3. 2; $\bar{2}$; 0; $\bar{4}$; $\bar{2}$; 0; 3. 4. $\cdot 312936$.
5. $1\cdot 32057$; $5\cdot 88453$; $\cdot\cdot 461791$.
6. (1) 21; (2) 13; (3) 30; (4) the 7th; (5) the 21st; (6) the 32nd.
7. (1) $\frac{4b}{c-b-a}$; (2) $\frac{a+2b}{4c-3b-2a}$; (3) $\frac{4a+7b}{a+3b-2c}$;
(4) $\frac{2b(2a-b)}{5ab+3ac-2b^2-bc}$ and $\frac{2ab}{5ab+3ac-2b^2-bc}$,
where $a = \log 2$, $b = \log 3$, and $c = \log 7$.
8. $\cdot 22221$. 9. $8\cdot 6415$. 10. $9\cdot 6192$.
11. $1\cdot 6389$. 12. $4\cdot 7162$. 13. $\cdot 41431$.

XXIV. (Pages 124-125)

1. $4\cdot 5527375$; $1\cdot 5527394$.
2. $4\cdot 7689529$; $\bar{3}\cdot 7689502$.
3. $478\cdot 475$; $\cdot 004784777$. 4. $2\cdot 583674$; $\cdot 0258362$.
5. (1) $4\cdot 7204815$; (2) $2\cdot 7220462$; (3) $\bar{4}\cdot 7240079$;
(4) $5273\cdot 63$; (5) $\cdot 05296726$; (6) $5\cdot 26064$.
6. $\cdot 6870417$. 7. 43° 23′ 45″
8. $\cdot 8455104$; $\cdot 8454509$. 9. 32° 16′35″; 32° 16′ 21″.
10. $4\cdot 1203060$; $4\cdot 1218748$.
11. $4\cdot 3993263$; $4\cdot 3976823$. 12. 13° 8′ 47″.
13. $9\cdot 9147334$. 14. 34° 44′ 27″.
15. $9\cdot 5254497$; 71° 27′ 43″. 16. $10\cdot 0229414$.
17. 18° 27′ 17″. 18. 36° 52′ 12″.

XXV. (Pages 127)

1. 13° 27′ 31″. 2. 22° 1′ 28″.

3. 1·0997340; 65° 24′ 12·5″.
4. 9·6198509; 22° 36′ 28″.
5. 10° 15′ 34″. 6. 44° 55′ 55″.
7. (1) 9·7279043; (2) 9·9270857; (3) 10·1958917;
   (4) 10·0757907; (5) 10·2061337;
   (6) 10·0725027; (7) 9·7245162.
8. (1) 57° 30′ 24″; (2) 57° 31′ 58″; (3) 32° 31′ 15″; (4) 57° 6′ 39″.
9. ·5373602.
10. (1) $\cos(x-y)\sec x \sec y$; (2) $\cos(x+y)\sec x \sec y$;
    (3) $\cos(x-y)\operatorname{cosec} x \sec y$; (4) $\cos(x+y)\operatorname{cosec} x \sec y$;
    (5) $\tan^2 x$; (6) $\tan x \tan y$.

XXVI. (Page 132-133)

1. $\frac{1}{5}$, $\frac{1}{2}$, and $\frac{9}{7}$.
2. $\frac{4}{\sqrt{41}}$, $\frac{3}{5}$ and $\frac{8}{5\sqrt{41}}$; $\frac{40}{41}$, $\frac{24}{25}$, and $\frac{496}{1025}$.
3. $\frac{3}{5}$, $\frac{4}{5}$ and 1.
4. $\frac{5}{12}$, $\frac{12}{5}$, and ∞. 5. $\frac{4}{5}$, $\frac{56}{65}$, and $\frac{12}{13}$.
6. $\frac{7}{41}$ and $\frac{287}{816}$. 7. 60°, 45°, and 75°.

XXVII. (Pages 136–138.)

23. 16·8 cm. 25. $\frac{2}{5}$. 28. $\frac{313}{338}$.

XXVIII. (Page 140.)

1. 186·60... and 193·18.
2. 26° 33′ 54″; 63° 26′ 6″; $10\sqrt{5}$ cm.
3. 48° 35′ 25″, 36° 52′ 12″, and 94° 32′ 23″.
4. 75° and 15°.

XXIX. (Pages 142.)

1. 90°. 2. 30°. 4. 120°.
5. 45°, 120°, and 15°. 6. 45°, 60° and 75°.
7. 58° 59′ 33″. 8. 77° 19′ 11″. 9. 76° 39′ 5″.
10. 104° 28′ 33″.
11. 56° 15′ 4″, 59° 51′ 10″, and 63° 53′ 46″.
12. 38° 56′ 33″, 47° 41′ 7″ and 93° 22′ 20″
13. 130° 42′ 20·5″, 23° 27′8·5″, and 25° 50′ 31″.

XXX. (Pages 146–147.)

1. 63° 13′ 2″; 43° 58′ 28″.
2. 117° 38′ 45″; 27° 38′ 45″.
3. $8\sqrt{7}$ cm; 79° 6′ 24″; 60°; 40° 53′ 36″.
4. 87° 27′ 25·5″; 32° 32′ 34·5″.
5. 40° 53′ 36″; 19° 6′ 24″; $\sqrt{7} : 2$.
6. 71° 44′ 30″; 48° 15′ 30″
7. 78° 17′ 40″; 49° 36′ 20″.
8. 108° 12′ 26″; 49° 27′ 34″.
9. $A = 45°$; $B = 75°$; $c = \sqrt{6}$.
10. $\sqrt{6}$; 15°; 105°.
11. ·8965.
14. 40 m.; 120°; 30°.
15. 7·589467; 108° 26′ 6″; 18° 26′ 6″; 53° 7′ 48″.
16. 2·529823.
17. 226·87; 73° 34′ 50″; 39° 45′ 10″
18. $A = 83° 7′ 39″$; $B = 42° 16′ 21″$; $c = 199·099$.
19. $B = 110° 48′ 15″$; $C = 26° 56′ 15″$; $a = 93·5192$
20. 73° 1′ 51″ and 48° 41′ 9″.
21. 88° 30′ 1″ and 33° 30′ 59″.

XXXI. (Pages 151-152)

1. There is no triangle.
2. $B_1 = 30°$, $C_1 = 105°$, and $b_1 = \sqrt{2}$; $B_2 = 60°$, $C_2 = 75°$ and $b_2 = \sqrt{6}$.
3. $B_1 = 15°$, $C_1 = 135°$, and $b_1 = 50(\sqrt{6} - \sqrt{2}) = 51·76$;
   $B_2 = 105°$, $C_2 = 45°$ and $b_2 = 50(\sqrt{6} + \sqrt{2}) = 193·185$.
5. $4\sqrt{3} \pm 2\sqrt{5}$, *i.e.* 11·4 and 2·46.
6. $100\sqrt{3}$; the triangle is right-angled.
8. 34° 27′ and 100° 33′.
9. 17·1 and 3·68.
10. (1) The triangle is right-angled and $B = 60°$.
    (2) $b_1 = 60·3893$, $B_1 = 8° 41′$ and $C_1 = 141° 19′$;
    $B_2 = 111° 19′$ and $C_2 = 38° 41′$.
11. 65° 59′ and 41° 56′ 12″.
12. 5·988... and 2·67... km. per hour.
13. 63° 2′ 12″ or 116° 57′ 48″.
14. 62° 31′ 23″ and 102° 17′ 37″, or 117° 28′ 37″ and 47° 20′ 23″.
15. 5926·61.

XXXII. (Page 153-154)

1. 7 : 9 : 11.
4. 79·063.
5. 1 km; 1·2197... km.
7. 20·976... km.
8. 6·857 and 5·438 cm.
9. 404·430 metres.
10. 233·2883 metres.
11. 2·229 metres

XXXIII. (Pages 157-159)

1. 100 m. high and 50 m. broad; 25 metres
2. 25·78... m.
3. 33·07... m.; $17\frac{1}{2}$ m.

4. 18·3... m. **5.** 12 metres. **6.** $h \tan\alpha \cot\beta$.
7. 1939·2... m. **8.** 10 metres **9.** 61·22... metres.
10. $100\sqrt[4]{2}$ (= 118·9) m.
15. $PQ = BP = BQ = 1000$ m.;
$AP = 500(\sqrt{6} - \sqrt{2}) = 517{\cdot}6$ m.; $AQ = 1000\sqrt{2} = 1414{\cdot}2$ m.
16 321·2 metres. **17.** 173·65 m; 984·81 m.
18. 119·286 m. **19.** 132·266 m.
20. 235·8 m. **21.** 1·4277 km.
22. 125·32 m.

XXXIV. (Pages 162-167)

3. 10 m; 20m.

4. l cosec γ, where γ is the sun's altitude; $\sin\gamma = \frac{2}{7}$, so that $\gamma = 16° 36'$.

5. 3·732... km; 12·342... km per hour at an angle, whose tangent is $\sqrt{3} + 1$, *i.e.* 69° 54′ S. of E.
6. 10·24... km per hour.
7. 16·39... km; 14·697... km.
8. 2·39 km; 1·366 km.

9. The angle whose tangent is $\frac{2}{3}$, *i.e.* 33° 41′; $\frac{9}{52}$ hour.

13. $c \sin\beta$ cosec $(\alpha + \beta)$; $c \sin\alpha \sin\beta$ cosec $(\alpha + \beta)$.

14. 9 m.; 2 m. **16.** $\frac{a}{3}, \frac{2a}{3}$.

20. At a distance $\frac{375}{\sqrt{7}}$ (= 141·74) m. from the cliff.

21. $c(1 - \sin\alpha)\sec\alpha$. **22.** 114·4123 m. **24.** 1069·75 m.

26. The angle whose tangent is $\frac{1}{2}$, *i.e.* 26° 34′. **29.** 45°.

32. 18° 26′ 6″. **34.** $\tan\alpha \sec\beta : 1$.
37. 9·1896 m. **38.** 1960·95 m.
39. 2·45832 km. **40.** 33·349 m.
41. 3·88 km. **42.** 492·4 and 459·17 metres.
43. 1·438 km. **45.** 251·13 metres per minute.

XXXV. (Pages 168, 169)

1. 84. **2.** 216. **3.** 630. **4.** 3720.
5. 270. **6.** 117096. **7.** 1470.
8. 1·183... **12.** 35 m. and 26 m.
13. 14·9... cm. **14.** 5, 7, and 8 cm. **15.** 120°.
17. 45° and 105°; 135° and 15°. **18.** 17·1064... sq. cm.

XXXVI. (Page 174-175)

3. $8\frac{1}{8}$, $1\frac{1}{2}$, 8, 2 and 24 respectively.

XXXVII. (Pages 180—183.)

35. 2·1547... or ·1547 times the radius of each circle.

39. $A_n = \frac{\pi}{3} + (-1)^n \cdot 2^n \cdot \left(A - \frac{\pi}{3}\right)$, ...

40. 61° 59′ 36″; 8·83 m.

XXXVIII. (Pages 187-188)

1. (1) $3\sqrt{105}$ (= 30·74) sq. m.; (2) $10\sqrt{7}$ (= 26·46) sq. m.

3. $1\frac{5}{7}$ and $2\frac{1}{2}$ m.

XXXIX. (Pages 190-191)

1. 77·98 cm. **2.** ·5359.

3. (1) 1·720... sq. m. (2) 2·598... sq. m.; (3) 4·8284... sq. m.; (4) 7·694... sq. m.; (5) 11·196... sq. m.

4. 1·8866... sq. cm. **5.** 3·3136... sq. cm.

6. $2 + \sqrt{2} : 4$; $\sqrt{2 + \sqrt{2}} : 2$. **12.** 3.

14. 6. **15.** 9. **16.** 20 and 10.

17. 6 and 5, 12 and 8, 18 and 10, 22 and 11, 27 and 12, 42 and 14, 54 and 15, 72 and 16, 102 and 17, 162 and 18, 342 and 19 sides respectively.

19. $\frac{2}{3}\sqrt{3}$; $\sqrt{6}$.

XL. (Pages 196-197)

1. ·00204. **2.** ·00007. **3.** ·00029.

4. ·99999. **5.** 25783·10077. **6.** 1·0000011.

7. 34′ 23″. **8.** 28° 40′ 37″. **9.** 39′ 42″.

10. 2° 33′ 44″. **11.** 114·59... cm.

XLI. (Pages 198)

1. 435·77 sq. m. **2.** 4·9087... sq. cm.

3. 127° 19′ 26″. **4.** 6 sq. cm.

5. 13·8 cm. **6.** 0·0007855 cm. **7.** $\frac{2}{3}\pi r$.

XLII. (Page 200)

1. 1° 48′ 40″. **2.** 2·53 km.

3. 974·77 m; 1° 26′ nearly.

8. 62 km nearly.

XLIII. (Pages 205–208)

28. $\pm\sqrt{\sin 2\beta}$. **29.** $\frac{1}{6}$ **30.** $\pm\frac{1}{\sqrt{2}}$. **31.** $4\sqrt{\frac{3}{7}}$.

32. $\frac{1}{4}$. 33. $n\pi$ or $n\pi + \frac{\pi}{4}$. 34. $\sqrt{3}$.

35. $\frac{\sqrt{5}}{3}$. 36. $\sqrt{3}$ or $-(2+\sqrt{3})$. 37. $\sqrt{3}$ or $2-\sqrt{3}$.

38. n or $n^2 - n + 1$. 39. $\frac{1}{2}\sqrt{\frac{3}{7}}$. 40. 13.

41. x is given by the equation.

$x^4 - x^2(ab + ac + ad + bc + bd + cd) + abcd = 0$.

42. $x = ab$.

43. $ab \div [\sqrt{a^2-1} - \sqrt{b^2-1}]$. 44. $\frac{a-b}{1+ab}$.

XLIV. (Pages 214–215)

1. $\frac{1}{2}\sin 2n\theta \operatorname{cosec}\theta$.

2. $\cos\frac{3n-1}{4}A \sin\frac{3n}{4}A \operatorname{cosec}\frac{3}{4}A$. 6. $\frac{1}{2}$.

7. $\sin\left[\alpha + \left(n - \frac{1}{2}\right)\beta\right] \sin n\beta \sec\frac{\beta}{2}$. 8. $-\sin\frac{n\theta}{n-2}$.

9. $\sin 2nx\,(\cos 2nx + \sin 2nx)(\cos x + \sin x)\operatorname{cosec} 2x$.

10. $\frac{1}{4}[(n+1)\sin 2\alpha - \sin(2n+2)\alpha]\operatorname{cosec}\alpha$.

11. $\frac{1}{2}\sin(2n+2)\alpha \,.\, \sin 2n\alpha \operatorname{cosec}\alpha$.

12. $\frac{n}{2}\cos 2\alpha - \frac{1}{2}\cos(n+3)\alpha \sin n\alpha \operatorname{cosec}\alpha$.

13. $\frac{\cos(2n\alpha - \alpha)\cos(n+1)\beta - \cos(2n\alpha + \alpha)\cos n\beta + \cos\alpha(1-\cos\beta)]}{2(\cos\beta - \cos 2\alpha)}$

14. $\frac{1}{4}[(2n+1)\sin\alpha - \sin(2n+1)\alpha]\operatorname{cosec}\alpha$.

15. $\frac{n}{2} - \frac{1}{2}\cos[2\theta + (n-1)\alpha]\sin n\alpha \operatorname{cosec}\alpha$.

16. $\frac{3}{4}\sin\frac{n+1}{2}a \sin\frac{na}{2}\operatorname{cosec}\frac{a}{2} - \frac{1}{4}\sin 3\frac{n+1}{2}a \sin\frac{3na}{2}\operatorname{cosec}\frac{3a}{2}$.

17. $\frac{1}{8}[3n - 4\cos(n+1)a \sin na \operatorname{cosec} a + \cos(2n+2)a \sin 2na \operatorname{cosec} 2a]$.

18. $\frac{1}{8}[3n + 4\cos(n+1)a \sin na \operatorname{cosec} a + \cos(2n+2)a \sin 2na \operatorname{cosec} 2a]$.

19. $\frac{1}{4}\sin\frac{n\theta}{2}\left[\cos\frac{n-1}{2}\theta + \cos\frac{n+3}{2}\theta + \cos\frac{n+7}{2}\theta\right]\operatorname{cosec}\frac{\theta}{2}$

$+ \frac{1}{4}\sin\frac{3n\theta}{2}\cos\frac{3n+9}{2}\theta \operatorname{cosec}\frac{3\theta}{2}$.

20. $-\frac{1}{2}\sin(2\alpha + 2n\beta)\sin 2n\beta \sec\beta.$

XLV. (Pages 218–219)

1. $a^2 + b^2 = c^2 + d^2.$
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos(\alpha - \beta) = \sin^2(\alpha - \beta).$
3. $a(2c^2 - d^2) = bcd.$
4. $a\sin\alpha + b\cos\alpha = \sqrt{2b(a+b)}.$
5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$
6. $\frac{x^2}{a} + \frac{y^2}{b} = a + b.$
7. $(p^2+1)^2 + 2q(p^2+1)(p+q) = 4(p+q)^2.$
8. $(x^2 + y^2 - b^2)^2 = a^2[(x+b)^2 + y^2].$
11. $a^2 + b^2 = 2 + 2\cos\alpha.$
12. $xy = (y - x)\tan\alpha.$
13. $a^2(a-c)(a-d) = b^2(b-c)(b-d).$
14. $8bc = a\{4b^2 + (b^2 - c^2)^2\}.$
15. $x(c^2 - a^2 - b^2) = y\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$
16. $b^2[x(b^2 - a^2) + a(a^2 + b^2)]^2 = 4a^4[b^2x^2 + a^2y^2].$

MISCELLANEOUS EXAMPLES. (Pages 223–246)

4. 267 m. approx.; 4°30′ approx.
5. 41, 50, and 21 cm.
6. $\sin(\beta - \alpha) = \pm\sqrt{1-b}\sqrt{1-a^2} \mp a\sqrt{b}.$
9. $n\pi + 38°\,7'\,27''.$
10. $\left(n + \frac{1}{4}\right)\frac{\pi}{3}.$
14. $7 - 3\sqrt{5} : 2.$
21. (i) $\theta = n\pi + (-1)^n\frac{\alpha+\beta}{2},$

    or $\tan\theta = (1 - \operatorname{cosec}\alpha\operatorname{cosec}\beta)\tan\frac{\alpha+\beta}{2};$

    (ii) $\theta = n\pi$ or $\left(n \pm \frac{1}{3}\right)\frac{\pi}{3}.$
22. 51° 19′; 78° 28′; 108° 13′.
23. 1298 metres nearly; 13° 13′ east of south.
28. 80 metres.

30. $\frac{1}{2}\tan^{-1}x;\ \frac{x+y}{1-xy}$.

31. $(l^2+m^2)(1-n)=2m(1+n)$.

32. Two roots.

35. $\left(m\pm\frac{1}{12}\right)\pi;\ \left(n\pm\frac{1}{6}\right)\pi$.

47. $(\lambda^2-1)^3=27\lambda^2\cos^2\alpha\sin^2\alpha$

48. 1·39 radians = 79°30′ nearly.

49. $\sin^2(\beta-\gamma)\sec^2(\alpha-\beta)\sec^2(\alpha-\gamma)$.

52. 11° 12′ north of east.

54. Six values

58. $\frac{1}{3}\left[n\pi+\frac{\pi}{2}-\alpha-\beta-\gamma\right]$.

59. 72·77 cm.

62. $c\sqrt{2a-b}=a\sqrt{2a}-(a-b)\sqrt{b}$.

63. $118\frac{1}{2}°$.

64. $-1°\,19'$, $+28\frac{1}{2}'$, and $+50\frac{1}{2}'$ nearly.

69. $\frac{a\sin\alpha\sin\beta}{\sqrt{\sin(\beta-\alpha)\sin(\beta+\alpha)}}$.

73. $\cos(\alpha+\beta+\gamma+\delta)+\cos(\alpha+\beta-\gamma-\delta)$
$+\cos(\alpha-\beta+\gamma-\delta)+\cos(\alpha-\beta-\gamma+\delta)-\cos(-\alpha+\beta+\gamma+\delta)$
$-\cos(\alpha-\beta+\gamma+\delta)-\cos(\alpha+\beta-\gamma+\delta)-\cos(\alpha+\beta+\gamma-\delta)$.

76. $66°\,19\frac{1}{2}'$ approx.

80. $x=-2\cdot4531$.

82. $\tan\theta=2\pm\sqrt{11}$ or $2\pm\sqrt{3}$; $\tan\phi=2\mp\sqrt{11}$ or $2\mp\sqrt{3}$.

84. 16·47 km.

87. $27y^2=x^2(9-8x^2)^2$.

94. $2a^3+c=3a(1+b)$.

95. $\sin\frac{n\theta}{2}\sin\frac{(n+3)\theta}{2}[1+2\cos2\theta]\operatorname{cosec}\frac{\theta}{2}$
$-\sin\frac{3n+9}{2}\theta\sin\frac{3n\theta}{2}\operatorname{cosec}\frac{3\theta}{2}$.

96. 2·55 radians = 146° 6′ nearly.

104. $\tan\theta=0,\ 1,\ -1,\ -2$.

108. 2·379 km.

117. 11°27′.

120. $x^2+y^2=a^2+b^2$.

128. $(a^2+b^2)^2-4a^2=\frac{8ab}{c}$.

136. $(x+y)^{2/5}+(x-y)^{2/5}=2$.

138. $\dfrac{mc_1 + na_1 + \sqrt{m^2c_2^2 + n^2a_2^2 + 2mn\, a_2 c_2 \cos(\alpha - \gamma)}\,\sin(nt + \beta)}{m + n}$.

where $\tan\beta = (mc_2 \sin\gamma + na_2 \sin\alpha) \div (mc_2 \cos\gamma + na_2 \cos\alpha)$.

146. $m^2 + m\cos\alpha = 2$.

148. $77\frac{1}{3}°; 63\frac{1}{2}°$.

151. $(x^2 + y^2 + 2ax)^2 = 4a^2(x^2 + y^2)$.

161. $(1 - x^2 - y^2)\sin(\alpha - \delta)\sin(\beta - \gamma)$.

176. $a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 = 0$.

177. $x^2y^2 - 4ty = (x + y)^2 \tan^2(\beta - \alpha)$.

178. $b^2(\sqrt{3}\,x - y)^2\{6(x^2 - y^2) - b^2\} = 8(x^2 - y^2)^3 + 4b^2(x^2 - y^2)$

181. 3 roots; 299° approx.

182. ·98 radian = 56° 9′ nearly.

185. 2·07 and − 1·23.

206. 48° 13′ 31″ E. of N. ; 24° 7′ 18″ W. of S.

CONSTANTS

One Radian = 57° 17′ 45″ nearly = 206265″;
log 206265 = 5·3144255.

$\pi = 3\cdot14159265.$	$\log\pi = 0\cdot4971499.$
$\frac{1}{\pi} = 0\cdot31830989.$	$\log\frac{1}{\pi} = \bar{1}\cdot5028501.$
$\frac{\pi}{180} = 0\cdot01745329.$	$\log\frac{\pi}{180} = \bar{2}\cdot2418774$
$\frac{180}{\pi} = 57\cdot2957795.$	$\log\frac{180}{\pi} = 1\cdot7581226.$
$\pi^2 = 9\cdot86960440.$	$\log\pi^2 = 0\cdot9942997.$
$\frac{1}{\pi^2} = 0\cdot10132118.$	$\log\frac{1}{\pi^2} = \bar{1}\cdot0057003.$
$\sqrt{\pi} = 1\cdot77245385.$	$\log\sqrt{\pi} = 0\cdot2485749.$
$\frac{1}{\sqrt{\pi}} = 0\cdot56418958.$	$\log\frac{1}{\sqrt{\pi}} = \bar{1}\cdot7514251.$
$\sqrt[3]{\pi} = 1\cdot46459189.$	$\log\sqrt[3]{\pi} = 0\cdot1657166.$
$\frac{1}{\sqrt[3]{\pi}} = 0\cdot68278406.$	$\log\frac{1}{\sqrt[3]{\pi}} = \bar{1}\cdot8342834.$
$\sqrt{2} = 1\cdot4142135...$	$\sqrt{3} = 1\cdot7320508...$
$\sqrt{5} = 2\cdot2360679...$	$\sqrt{6} = 2\cdot4494897...$
$\sqrt{7} = 2\cdot6457513...$	$\sqrt{8} = 2\cdot8284271...$
$\sqrt{10} = 3\cdot1622776...$	

TABLES OF LOGARITHMS
NATURAL SINES, COSINES AND TANGENTS,
LOGARITHMIC SINES, COSINES TANGENTS
AND RADIAN MEASURE OF ANGLES

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
10	00000	00432	00860	01284	01703						42	85	127	170	212	254	297	339	381
						02119	02531	02938	03342	03743	40	81	121	162	202	242	283	323	364
11	04139	04532	04922	05308	05690						37	77	116	154	193	232	270	309	348
						06070	06446	06819	07188	07555	37	74	111	148	185	222	259	296	333
12	07918	08279	08636	08991	09342						36	71	106	142	177	213	248	284	319
						09691	10037	10380	10721	11059	34	68	102	136	170	204	238	272	307
13	11394	11727	12057	12385	12710						33	66	98	131	164	197	229	262	295
						13033	13354	13672	13988	14301	32	63	95	126	158	190	221	253	284
14	14613	14922	15229	15534	15836						30	61	91	122	152	183	213	244	274
						16137	16435	16732	17026	17319	29	59	88	118	147	177	206	236	265
15	17609	17898	18184	18469	18752						28	57	85	114	142	171	199	228	256
						19033	19312	19590	19866	20140	28	55	83	110	138	165	193	221	248
16	20412	20683	20951	21219	21484						27	53	80	107	134	160	187	214	240
						21748	22011	22272	22531	22789	26	52	78	104	130	156	182	208	233
17	23045	23300	23553	23805	24055						26	50	76	101	126	151	176	201	227
						24304	24551	24797	25042	25285	25	49	73	98	122	147	171	196	220
18	25527	25768	26007	26245	26482						24	48	71	95	119	143	167	190	214
						26717	26951	27184	27416	27646	23	46	69	93	116	139	162	185	208
19	27875	28103	28330	28556	28780						23	45	68	90	113	135	158	180	203
						29003	29226	29447	29667	29885	22	44	66	88	110	132	154	176	198
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015	21	43	64	85	106	127	148	170	190
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044	20	41	61	81	101	121	141	162	182
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984	20	39	58	77	97	116	135	154	174
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840	19	37	56	74	93	111	130	148	167
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620	18	35	53	71	89	106	124	142	159
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330	17	34	51	68	35	102	119	136	153
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975	16	33	49	66	82	98	115	131	148
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560	16	32	47	63	79	95	111	126	142
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090	15	30	46	61	76	91	107	122	137
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567	15	29	44	59	74	88	103	118	132
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996	14	29	43	57	72	86	100	114	129
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379	14	28	41	55	69	83	97	110	124
32	50515	50650	50786	50920	51054	51188	51322	51455	51587	51720	13	27	40	54	67	80	94	107	121
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020	13	26	39	52	65	78	91	104	117
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283	13	25	38	50	63	76	88	101	113
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509	12	24	37	49	61	73	85	98	110
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703	12	24	36	48	60	71	83	95	107
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864	12	23	35	46	58	70	81	93	104
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995	11	23	34	45	57	68	79	90	102
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097	11	22	33	44	55	66	77	88	99
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172	11	21	32	43	54	64	75	86	97
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221	10	21	31	42	53	63	74	84	95
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246	10	20	31	41	51	61	71	82	92
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246	10	20	30	40	50	60	70	80	90
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225	10	20	29	39	49	59	68	78	88
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181	10	19	29	38	48	57	67	76	86
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117	9	19	28	37	47	56	65	74	84
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034	9	18	27	36	46	55	64	73	82
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931	9	18	27	36	45	53	63	72	81
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810	9	18	26	35	44	53	62	70	79

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672	9	17	26	34	43	52	60	69	77
51	70757	70842	70927	71013	71096	71181	71265	71349	71433	71517	8	17	25	34	42	50	59	67	76
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346	8	17	25	33	42	50	58	66	75
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159	8	16	24	32	41	49	57	65	73
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957	8	16	24	32	40	48	56	64	72
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	8	16	23	31	39	47	55	63	70
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	8	15	23	31	39	46	54	62	69
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	8	15	23	30	38	45	53	60	68
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012	7	15	22	30	37	44	52	59	67
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	7	15	22	29	37	44	51	58	66
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	7	14	22	29	36	43	50	58	65
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	7	14	21	28	36	43	50	57	64
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	7	14	21	28	35	41	48	55	62
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	7	14	20	27	34	41	48	54	61
64	80618	80685	80754	80821	80889	80956	81023	81090	81158	81244	7	13	20	27	34	41	47	54	60
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889	7	13	20	26	33	40	46	53	59
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543	7	13	20	26	33	39	46	52	59
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187	6	13	19	26	32	38	45	51	58
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822	6	13	19	25	32	38	44	50	57
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448	6	12	19	25	31	37	43	50	56
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065	6	12	19	25	31	37	43	50	56
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673	6	12	18	24	31	37	43	49	55
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273	6	12	18	24	30	36	42	48	54
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864	6	12	18	24	30	35	41	47	53
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448	6	12	17	23	29	35	41	46	52
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024	6	12	17	23	29	35	41	46	52
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593	6	11	17	23	29	34	40	46	51
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154	6	11	17	22	28	34	39	45	50
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708	6	11	17	22	28	33	39	44	50
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255	6	11	17	22	28	33	39	44	50
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795	5	11	16	22	27	32	38	43	49
81	90848	90902	90956	91009	91062	91116	91169	91222	91275	91328	5	11	16	21	27	32	37	42	48
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855	5	11	16	21	27	32	37	42	48
83	91908	91960	92012	92064	92117	92169	92221	92273	92324	92376	5	10	16	21	26	31	36	42	47
84	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891	5	10	15	20	26	31	36	41	46
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399	5	10	15	20	26	31	36	41	46
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902	5	10	15	20	25	30	35	40	45
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399	5	10	15	20	25	30	35	40	45
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890	5	10	15	20	25	29	34	39	44
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376	5	10	15	19	24	29	34	39	44
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856	5	10	14	19	24	29	34	38	43
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332	5	9	14	19	24	28	33	38	42
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802	5	9	14	19	24	28	33	38	42
93	96848	96895	96942	69688	97035	97081	97128	97174	97220	97267	5	9	14	18	23	28	32	38	42
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727	5	9	14	18	23	28	32	37	42
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182	5	9	14	18	23	27	32	36	41
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632	5	9	14	18	23	27	32	36	41
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078	4	9	13	18	22	27	31	36	40
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520	4	9	13	18	22	26	31	35	40
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957	4	9	13	17	22	26	31	35	39

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
.00	10000	10023	10046	10069	10093	10116	10139	10162	10186	10209	2	5	7	9	12	14	16	19	21
.01	10233	10257	10280	10304	10328	10351	10375	10399	10423	10447	2	5	7	10	12	14	17	19	21
.02	10471	10495	10520	10544	10568	10593	10617	10641	10666	10691	2	5	7	10	12	15	17	20	22
.03	10715	10740	10765	10789	10814	10839	10864	10889	10914	10940	3	5	8	10	13	15	18	20	23
.04	10965	10990	11015	11041	11066	11092	11117	11143	11169	11194	3	5	8	10	13	15	18	20	23
.05	11220	11246	11272	11298	11324	11350	11376	11402	11429	11455	3	5	8	11	13	16	18	21	24
.06	11482	11508	11535	11561	11588	11614	11641	11668	11695	11722	3	5	8	11	13	16	19	21	24
.07	11749	11776	11803	11830	11858	11885	11912	11940	11967	11995	3	5	8	11	14	16	19	22	25
.08	12023	12050	12078	12106	12134	12162	12190	12218	12246	12274	3	6	8	11	14	17	20	22	25
.09	12303	12331	12359	12388	12417	12445	12474	12503	12531	12560	3	6	9	11	14	17	20	23	26
.10	12589	12618	12647	12677	12706	12735	12764	12794	12823	12853	3	6	9	12	15	18	21	24	26
.11	12882	12912	12942	12972	13002	13032	13062	13092	13122	13152	3	6	9	12	15	18	21	24	27
.12	13183	13213	13243	13274	13305	13335	13366	13397	13428	13459	3	6	9	12	15	18	21	25	28
.13	13490	13521	13552	13583	13614	13646	13677	13709	13740	13772	3	6	9	13	16	19	22	25	28
.14	13804	13836	13868	13900	13932	13964	13996	14028	14060	14093	3	6	10	13	16	19	22	26	29
.15	14125	14158	14191	14223	14256	14289	14322	14355	14388	14421	3	7	10	13	16	20	23	26	30
.16	14454	14488	14521	14555	14588	14622	14655	14689	14723	14757	3	7	10	13	17	20	24	27	30
.17	14791	14825	14859	14894	14928	14962	14997	15031	15066	15101	3	7	10	14	17	21	24	28	31
.18	15136	15171	15205	15241	15276	15311	15346	15382	15417	15453	4	7	11	14	18	21	25	28	32
.19	15488	15524	15560	15596	15631	15668	15704	15740	15776	15812	4	7	11	14	18	22	25	29	32
.20	15849	15885	15922	15959	15996	16032	16069	16106	16144	16181	4	7	11	15	18	22	26	30	33
.21	16218	16255	16293	16331	16368	16406	16444	16482	16520	16558	4	8	11	15	19	23	26	30	34
.22	16596	16634	16672	16711	16749	16788	16827	16866	16904	16943	4	8	12	15	19	23	27	31	35
.23	16982	17022	17061	17100	17140	17179	17219	17258	17298	17338	4	8	12	16	20	24	28	32	36
.24	17378	17418	17458	17498	17539	17579	17620	17660	17701	17742	4	8	12	16	20	24	28	32	36
.25	17783	17824	17865	17906	17947	17989	18030	18072	18113	18155	4	8	12	17	21	25	29	33	37
.26	18197	18239	18281	18323	18365	18408	18450	18493	18535	18578	4	8	13	17	21	25	30	34	38
.27	18621	18664	18707	18750	18793	18836	18880	18923	18967	19011	4	9	13	17	22	26	30	35	39
.28	19055	19099	19143	19187	19231	19275	19320	19364	19409	19454	4	9	13	18	22	26	31	35	40
.29	19498	19543	19588	19634	19679	19724	19770	19815	19861	19907	5	9	14	18	23	27	32	36	41
.30	19953	19999	20045	20091	20137	20184	20230	20277	20324	20370	5	9	14	19	23	28	32	37	42
.31	20417	20464	20512	20559	20606	20654	20701	20749	20797	20845	5	10	14	19	24	29	33	38	43
.32	20893	20941	20989	21038	21086	21135	21184	21232	21281	21330	5	10	15	19	24	29	34	39	44
.33	21380	21429	21478	21528	21577	21627	21677	21727	21777	21827	5	10	15	20	25	30	35	40	45
.34	21878	21928	21979	22029	22080	22131	22182	22233	22284	22336	5	10	15	20	25	31	36	41	46
.35	22387	22439	22491	22542	22594	22646	22699	22751	22803	22856	5	10	16	21	26	31	37	42	47
.36	22909	22961	23014	23067	23121	23174	23227	23281	23336	23388	5	11	16	21	27	32	37	43	48
.37	23442	23496	23550	23605	23659	23714	23768	23823	23878	23933	5	11	16	22	27	33	38	44	49
.38	23988	24044	24099	24155	24210	24266	24322	24378	24434	24491	6	11	17	22	28	34	39	45	50
.39	24547	24604	24660	24717	24774	24831	24889	24946	25003	25061	6	11	17	23	29	34	40	46	51
.40	25119	25177	25236	25293	25351	25410	25468	25527	25586	25645	6	12	18	23	29	35	41	47	53
.41	25704	25763	25823	25882	25942	26002	26062	26122	26182	26242	6	12	18	24	30	36	42	48	54
.42	26303	26363	26424	26485	26546	26607	26669	26730	26792	26853	6	12	18	24	31	37	43	49	55
.43	26915	26977	27040	27102	27164	27227	27290	27353	27416	27479	6	13	19	25	31	38	44	50	56
.44	27542	27606	27669	27733	27797	27861	27925	27990	28054	28119	6	13	19	26	32	39	45	51	58
.45	28184	28249	28314	28379	28445	28510	28576	28642	28708	28774	7	13	20	26	33	39	46	52	59
.46	28840	28907	28973	29040	29107	29174	29242	29309	29376	29444	7	13	20	27	34	40	47	54	60
.47	29512	29580	29648	29717	29785	29854	29923	29992	30061	30130	7	14	21	28	34	41	48	55	62
.48	30200	30269	30339	30409	30479	30549	30620	30690	30761	30832	7	14	21	28	35	42	49	56	63
.49	30930	30974	31046	31117	31189	31261	31333	31405	31477	31550	7	14	22	29	36	43	50	58	65

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences.								
											1	2	3	4	5	6	7	8	9
.50	31623	31696	31769	31842	31916	31989	32063	32137	32211	32285	7	15	22	29	37	44	52	59	66
.51	32359	32434	32509	32584	32659	32735	32809	32885	32961	33037	8	15	23	30	38	45	53	60	68
.52	33884	33189	33266	33343	33420	33497	33574	33651	33729	33806	8	15	23	31	39	46	54	62	69
.53	33888	33963	34041	34119	34198	34277	34356	34435	34514	34594	8	16	24	32	40	47	55	63	71
.54	34674	34754	34834	34914	34995	35075	35156	35237	35318	35400	8	16	24	32	40	48	56	65	73
.55	35481	35563	35645	35727	35810	35892	35975	36058	36141	36224	8	16	25	33	41	50	58	66	74
.56	36308	36392	36475	36559	36644	36728	36813	36898	36983	37068	8	17	25	34	42	51	59	68	76
.57	37154	37239	37325	37411	37497	37584	37670	37757	37844	37931	9	17	26	35	43	52	61	69	78
.58	38019	38107	38194	38282	38371	38459	38548	38637	38726	38815	9	18	27	35	44	53	62	71	80
.59	38905	38994	39084	39174	39264	39355	39446	39537	39628	39719	9	18	27	36	45	54	63	72	82
.60	39811	39902	39994	40087	40179	40272	40365	40458	40551	40644	9	19	28	37	46	56	65	74	83
.61	40738	40832	40926	41020	41115	41210	41305	41400	41495	41591	9	19	28	38	47	57	66	76	85
.62	41687	41783	41879	41976	42073	42170	42267	42364	42462	42560	10	19	29	39	49	58	68	78	87
.63	42658	42756	42855	42954	43053	43152	43251	43351	43451	43551	10	20	30	40	50	60	70	80	89
.64	43652	43752	43853	43954	44055	44157	44259	44361	44463	44566	10	20	30	41	51	61	71	81	91
.65	44668	44771	44875	44978	45082	45186	45290	45394	45499	45604	10	21	31	42	52	62	73	83	94
.66	45709	45814	45920	46026	46132	46238	46345	46452	46559	46666	11	21	32	43	53	64	75	85	96
.67	46774	46881	46989	47098	47206	47315	47424	47534	47643	47753	11	22	33	44	54	65	76	87	98
.68	47863	47973	48084	48195	48306	48417	48529	48641	48753	48865	11	22	33	45	56	67	78	89	100
.69	48978	49091	49204	49317	49431	49545	49659	49774	49888	50003	11	23	34	46	57	68	80	91	103
.70	50119	50234	50350	50466	50582	50699	50816	50933	51050	51168	12	23	35	47	58	70	82	93	105
.71	51286	51404	51523	51642	51761	51880	52000	52119	52240	52360	12	24	36	48	60	72	84	96	108
.72	52481	52602	52723	52845	52966	53088	53211	53333	53456	53580	12	24	37	49	61	73	85	98	110
.73	53703	53827	53951	54075	54200	54325	54450	54576	54702	54828	13	25	38	50	63	75	88	100	113
.74	54954	55081	55208	55336	55463	55590	55719	55847	55976	56105	13	26	38	51	64	77	90	102	115
.75	56234	56364	56494	56624	56754	56885	57016	57148	57280	57412	13	26	39	52	66	79	92	105	118
.76	57544	57677	57810	57943	58076	58210	58345	58479	58614	58749	13	27	40	54	67	80	94	107	121
.77	58884	59020	59156	59293	59429	59566	59704	59841	59979	60117	14	27	41	55	69	82	96	110	123
.78	60256	60395	60534	60674	60814	60954	61094	61235	61376	61518	14	28	42	56	70	84	98	112	126
.79	61659	61802	61944	62087	62230	62373	62517	62661	62806	62951	14	29	43	58	72	86	101	115	130
.80	63096	63241	63387	63533	63680	63826	63973	64121	64269	64417	15	29	44	59	74	88	103	118	132
.81	64565	64714	64863	65013	65163	65313	65464	65615	65766	65917	15	30	45	60	75	90	105	120	135
.82	66069	66222	66374	66527	66681	66834	66988	67143	67298	67453	15	31	46	62	77	92	108	123	139
.83	67608	67764	67920	68077	68234	68391	68549	68707	68865	69024	16	32	47	63	79	95	110	126	142
.84	69183	69343	69503	69663	69823	69984	70146	70307	70469	70632	16	32	48	64	81	97	113	129	145
.85	70795	70958	71121	71285	71450	71614	71779	71945	72111	72277	17	33	50	66	83	99	116	132	149
.86	72444	72611	72778	72946	73114	73282	73451	73621	73790	73961	17	34	51	68	85	101	118	135	152
.87	74131	74302	74473	74645	74817	74989	75162	75336	75509	75683	17	35	52	69	87	104	121	138	156
.88	75858	76033	76208	76384	76560	76736	76913	77090	77268	77446	18	35	53	71	89	107	125	142	159
.89	77625	77804	77983	78163	78343	78524	78705	78886	79068	79250	18	36	54	72	91	109	127	145	163
.90	79433	79616	79799	79983	80168	80353	80538	80724	80910	81096	19	37	56	74	93	111	130	148	167
.91	81283	81470	81658	81846	82035	82224	82414	82604	82794	82985	19	38	57	76	95	113	132	151	170
.92	83176	83368	83560	83753	83946	84140	84333	84528	84723	84918	19	39	58	78	97	116	136	155	175
.93	85114	85310	85507	85704	85901	86099	86298	86497	86696	86896	20	40	60	79	99	119	139	158	178
.94	87096	87297	87498	87700	87902	88105	88308	88512	88716	88920	20	41	61	81	102	122	142	162	183
.95	89125	89331	89536	89743	89950	90157	90365	90573	90782	90991	21	42	62	83	104	125	146	166	187
.96	91201	91411	91622	91833	92045	92257	92470	92683	92897	93111	21	42	64	85	106	127	149	170	191
.97	93325	93541	93756	93972	94189	94406	94624	94842	95060	95280	22	43	65	87	109	130	152	174	195
.98	95499	95719	95940	96161	96383	96605	96828	97051	97275	97499	22	44	67	89	111	133	155	178	200
.99	97724	97949	98175	98401	98628	98855	99083	99311	99541	99770	23	46	68	91	114	137	160	182	205

NATURAL SINES.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3.	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	.00000	00175	00349	00524	00698	00873	01047	01222	01396	01571	29	58	87	117	145
1	.01745	01920	02094	02269	02443	02618	02792	02967	03141	03316	29	58	87	117	145
2	.03490	03664	03839	04013	04188	04362	04536	04711	04885	05059	29	58	87	116	145
3	.05234	05408	05582	05756	05931	06105	06279	06453	06627	06802	29	58	87	116	145
4	.06976	07150	07324	07498	07672	07846	08020	08194	08368	08542	29	58	87	116	145
5	.08716	08889	09063	09237	09411	09585	09758	09932	10106	10279	29	58	87	116	145
6	.10453	10626	10800	10973	11147	11320	11494	11667	11840	12014	29	58	87	116	145
7	.12187	12360	12533	12706	12880	13053	13226	13399	13572	13744	29	58	87	115	144
8	.13917	14090	14263	14436	14608	14781	14954	15126	15299	15471	29	58	86	115	144
9	.15643	15816	15988	16160	16333	16505	16677	16849	17021	17193	29	57	86	115	144
10	.17365	17537	17708	17880	18052	18224	18395	18567	18738	18910	29	57	86	114	143
11	.19081	19252	19423	19595	19766	19937	20108	20279	20450	20620	29	57	86	114	143
12	.20791	20962	21132	21303	21474	21644	21814	21985	22155	22325	28	57	85	114	142
13	.22495	22665	22835	23005	23175	23345	23514	23684	23853	24023	28	57	85	113	141
14	.24192	24362	24531	24700	24869	25038	25207	25376	25545	25713	28	56	85	113	141
15	.25882	26050	26219	26387	26556	26724	26892	27060	27228	27396	28	56	84	112	140
16	.27564	27731	27899	28067	28234	28402	28569	28736	28903	29070	28	56	84	112	139
17	.29237	29404	29571	29737	29904	30071	30237	30403	30570	30736	28	56	83	111	139
18	.30902	31068	31233	31399	31565	31730	31896	32061	32227	32392	28	55	83	110	138
19	.32557	32722	32887	33051	33216	33381	33545	33710	33874	34038	27	55	82	110	137
20	.34202	34366	34530	34694	34857	35021	35184	35347	35511	35674	27	55	82	109	136
21	.35837	36000	36162	36325	36488	36650	36812	36975	37137	37299	27	54	81	108	135
22	.37461	37622	37784	37946	38107	38268	38430	38591	38752	38912	27	54	81	107	134
23	.39073	39234	39394	39555	39715	39875	40035	40195	40355	40514	27	53	80	107	133
24	.40674	40833	40992	41151	41310	41469	41628	41787	41945	42104	27	53	79	106	132
25	.42262	42420	42578	42736	42894	43051	43209	43366	43523	43680	26	53	79	105	131
26	.43837	43994	44151	44307	44464	44620	44776	44932	45088	45243	26	52	78	104	130
27	.45399	45554	45710	45865	46020	46175	46330	46484	46639	46793	26	52	77	103	129
28	.46947	47101	47255	47409	47562	47716	47869	48022	48175	48328	26	51	77	102	128
29	.48481	48634	48786	48938	49090	49242	49394	49546	49697	49849	25	51	76	101	126
30	.50000	50151	50302	50453	50603	50754	50904	51054	51204	51354	25	50	75	100	125
31	.51504	51653	51803	51952	52101	52250	52399	52547	52696	52844	25	50	74	99	124
32	.52992	53140	53288	53435	53583	53730	53877	54024	54171	54317	25	49	74	98	123
33	.54464	54610	54756	54902	55048	55194	55339	55484	55630	55775	24	49	73	97	121
34	.55919	56064	56208	56353	56497	56641	56784	56928	57071	57215	24	48	72	96	120
35	.57358	57501	57643	57786	57928	58070	58212	58354	58496	58637	24	48	71	95	118
36	.58778	58920	59061	59201	59342	59482	59622	59763	59902	60042	23	47	70	94	117
37	.60181	60321	60460	60599	60738	60876	61015	61153	61291	61429	23	46	69	92	115
38	.61566	61704	61841	61978	62115	62251	62388	62524	62660	62796	23	46	68	91	114
39	.62932	63068	63203	63338	63473	63608	63742	63877	64011	64145	23	45	67	90	112
40	.64279	64412	64546	64679	64812	64945	65077	65210	65342	65474	22	44	66	88	111
41	.65606	65738	65869	66000	66131	66262	66393	66523	66653	66783	22	44	65	87	109
42	.66913	67043	67172	67301	67430	67559	67688	67816	67944	68072	22	43	64	86	107
43	.68200	68327	68455	68582	68709	68835	68962	69088	69214	69340	21	42	63	84	106
44	.69466	69591	69717	69842	69966	70091	70215	70339	70463	70587	21	41	62	83	104

NATURAL SINES.

Degrees	0′ 0°.0	6′ 0°.1	12′ 0°.2	18′ 0°.3	24′ 0°.4	30′ 0°.5	36′ 0°.6	42′ 0°.7	48′ 0°.8	54′ 0°.9	Mean Differences. 1	2	3	4	5
45	.70711	70834	70957	71080	71203	71325	71447	71569	71691	71813	20	41	61	82	102
46	.71934	72055	72176	72297	72417	72537	72657	72777	72897	73016	20	40	60	80	100
47	.73135	73254	73373	73491	73610	73728	73846	73963	74080	74198	20	39	59	79	98
48	.74314	74431	74548	74664	74780	74896	75011	75126	75241	75356	20	39	58	77	96
49	.75471	75585	75700	75813	75927	76041	76154	76267	76380	76492	19	38	57	76	94
50	.76604	76717	76828	76940	77051	77162	77273	77384	77494	77605	19	37	56	74	93
51	.77715	77824	77934	78043	78152	78261	78369	78478	78586	78694	18	36	54	72	91
52	.78801	78908	79015	79122	79229	79335	79547	79653	79758	79758	18	35	53	71	89
53	.79864	79968	80073	80178	80282	80386	80489	80593	80696	80799	17	35	52	69	87
54	.80902	81004	81106	81208	81310	81412	81513	81614	81714	81815	17	34	51	68	84
55	.81915	82015	82115	82214	82314	82413	82511	82610	82708	82806	16	33	49	66	82
56	.82904	83001	83098	83195	83292	83389	83485	83581	83676	83772	16	32	48	64	80
57	.83867	83962	84057	84151	84245	84339	84433	84526	84619	84712	16	31	47	62	78
58	.84805	84897	84989	85081	85173	85264	85355	85446	85536	85627	15	31	46	61	76
59	.85717	85806	85896	85985	86074	86163	86251	86340	86427	86515	15	30	44	59	74
60	.86603	86690	86777	86863	86949	87036	87121	87207	87292	87377	14	29	43	57	72
61	.87462	87546	87631	87715	87798	87882	87965	88048	88130	88213	14	28	42	56	69
62	.88295	88377	88458	88539	88620	88701	88782	88862	88942	89021	14	27	40	54	67
63	.89101	89180	89259	89337	89415	89493	89571	89649	89726	89803	13	26	39	52	65
64	.89879	89956	90032	90108	90183	90259	90334	90408	90483	90557	13	25	38	50	63
65	.90631	90704	90778	90851	90924	90996	91068	91140	91212	91283	12	24	36	48	60
66	.91355	91425	91496	91566	91636	91706	91775	91845	91914	91982	12	23	35	46	58
67	.92050	92119	92186	92254	92321	92388	92455	92521	92587	92653	11	22	33	45	56
68	.92718	92784	92849	92913	92978	93042	93106	93169	93232	93295	11	21	32	43	54
69	.93358	93420	93483	93544	93606	93667	93728	93789	93849	93909	10	20	31	41	51
70	.93969	94029	94088	94147	94206	94264	94322	94380	94438	94495	10	19	29	39	49
71	.94552	94609	94665	94721	94777	94832	94888	94943	94997	95052	9	18	28	37	46
72	.95106	95159	95213	95266	95319	95372	95424	95476	95528	95579	9	17	26	35	44
73	.95630	95681	95732	95782	95832	95882	95931	95981	96029	96078	8	17	25	33	41
74	.96126	96174	96222	96269	96316	96363	96410	96456	96502	96547	8	16	23	31	39
75	.96593	96638	96682	96727	96771	96815	96858	96902	96945	96987	7	16	22	29	36
76	.97030	97072	97113	97155	97196	97237	97278	97318	97358	97398	7	14	20	27	34
77	.97437	97476	97515	97553	97592	97630	97667	97705	97742	97778	6	13	19	25	32
78	.97815	97851	97887	97922	97958	97992	98027	98061	98096	98129	6	12	17	23	29
79	.98163	98196	98229	98261	98294	98325	98357	98388	98420	98450	5	11	16	21	27
80	.98481	98511	98541	98570	98600	98629	98657	98686	98714	98741	5	10	14	19	24
81	.98769	98796	98823	98849	98876	98902	98927	98953	98978	99002	4	9	13	17	22
82	.99027	99051	99075	99098	99122	91444	99167	99189	99211	99233	4	8	11	15	19
83	.99255	99276	99297	99317	99337	99357	99377	99396	99415	99434	3	7	10	13	16
84	.99452	99470	99488	99506	99523	99540	99556	99572	99588	99604	3	6	8	11	14
85	.99619	99635	99649	99664	99678	99692	99705	99719	99731	99744	2	5	7	9	11
86	.99756	99768	99780	99792	99803	99813	99824	99834	99844	99854	2	4	5	7	9
87	.99863	99872	99881	99889	99897	99905	99912	99919	99926	99933	1	3	4	5	6
88	.99939	99945	99951	99956	99961	99966	99970	99974	99978	99982	1	2	2	3	4
89	.99985	99988	99990	99993	99995	99996	99998	99999	99999	1.0000	0	1	1	1	1
90	1.000														

NATURAL COSINES.

Numbers in difference columns to be subtracted, not added.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3.	3°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	1.000	99999	99999	99999	99998	99996	99995	99993	99990	99988	0	1	1	1	1
1	.99985	99982	99978	99974	99970	99966	99961	99956	99951	99945	1	2	2	3	4
2	.99939	99933	99926	99919	99912	99905	99897	99889	99881	99872	1	3	4	5	6
3	.99863	99854	99844	99834	99824	99813	99803	99792	99780	99768	2	4	5	7	9
4	.99756	99744	99731	99719	99705	99692	99678	99664	99649	99635	2	5	7	9	11
5	.99619	99604	99588	99572	99556	99540	99523	99506	99488	99470	3	6	8	11	14
6	.99452	99434	99415	99396	99377	99357	99337	99317	99297	99276	3	7	10	13	16
7	.99255	99233	99211	99189	99167	99144	99122	99098	99075	99051	4	8	11	15	19
8	.99027	99002	98978	98953	98927	98902	98876	98849	98823	98796	4	9	13	17	22
9	.98769	98741	98714	98686	98657	98629	98600	98570	98541	98511	5	10	14	19	24
10	.98481	98450	98420	98388	98357	98325	98294	98261	98229	98196	5	11	16	21	27
11	.98163	98129	98096	98061	98027	97992	97958	97922	97887	97851	6	12	17	23	29
12	.97815	97778	97742	97705	97667	97630	97592	97553	97515	97476	6	13	19	25	32
13	.97437	97398	97358	97318	97278	97237	97196	97155	97113	97072	7	14	20	27	34
14	.97030	96987	96945	96902	96858	96815	96771	96727	96682	96638	7	16	22	29	36
15	.96593	96547	96502	96456	96410	96363	96316	96269	96222	96174	8	16	23	31	39
16	.96126	96078	96029	95981	95931	95882	95832	95782	95732	95681	8	17	25	33	41
17	.95630	95579	95528	95476	95424	95372	95319	95266	95213	95159	9	17	26	35	44
18	.95106	95052	94997	94943	94888	94832	94777	94721	94665	94609	9	18	28	37	46
19	.94552	94495	94438	94380	94322	94264	94206	94147	94088	94029	10	19	29	39	49
20	.93969	93909	93849	93789	93728	93667	93606	93544	93483	93420	10	20	31	41	51
21	.93358	93295	93232	93169	93106	93042	92978	92913	92849	92784	10	21	32	43	54
22	.92718	92653	92587	92521	92455	92388	92321	92254	92186	92119	11	22	33	45	56
23	.92050	91982	91914	91845	91775	91706	91636	91566	91496	91425	11	23	35	46	58
24	.91355	91283	91212	91140	91068	90996	90924	90851	90778	90704	12	24	36	48	60
25	.90631	90557	90483	90408	90334	90259	90183	90108	90032	89956	13	25	38	50	63
26	.89879	89803	89726	89649	89571	89493	89415	89337	89259	89180	13	26	39	52	65
27	.89101	89021	88942	88862	88782	88701	88620	88539	88458	88377	13	27	40	54	67
28	.88295	88213	88130	88048	87965	87882	87798	87715	87631	87546	14	28	42	56	69
29	.87462	87377	87292	87207	87121	87036	86949	86863	86777	86690	14	29	43	57	72
30	.86603	86515	86427	86340	86251	86163	86074	85985	85896	85806	15	30	44	59	74
31	.85717	85627	85536	85446	85355	85264	85173	85081	84989	84897	16	30	46	61	76
32	.84805	84712	84619	84526	84433	84339	84245	84151	84057	83962	16	31	47	62	78
33	.83867	83772	83676	83581	83485	83389	83292	83195	83098	83001	16	32	48	64	80
34	.82904	82806	82708	82610	82511	82413	82314	82214	82115	82015	16	33	49	66	82
35	.81915	81815	81714	81614	81513	81412	81310	81208	81106	81004	17	34	51	68	84
36	.80902	80799	80696	80593	80489	80386	80282	80178	80073	79968	17	35	52	69	87
37	.79864	79758	79653	79547	79441	79335	79229	79122	79015	78908	18	35	53	71	89
38	.78801	78693	78586	78478	78369	78261	78152	78043	77934	77824	18	36	54	72	91
39	.77715	77605	77494	77384	77273	77162	77051	76940	76828	76717	18	37	56	74	93
40	.76604	76492	76380	76267	76154	76041	75927	75813	75700	75585	19	38	57	76	94
41	.75471	75356	75241	75126	75011	74896	74780	74664	74548	74431	19	39	58	77	96
42	.74314	74198	74080	73963	73846	73728	73610	73491	73373	73254	20	39	59	79	98
43	.73135	73016	72897	72777	72657	72537	72417	72297	72176	72055	20	40	60	80	100
44	.71934	71813	71691	71569	71447	71325	71203	71080	70957	70834	20	41	61	82	102

NATURAL COSINES.

Numbers in difference columns to be subtracted, not added.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3.	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	.70711	70587	70463	70339	70215	70091	69966	69842	69717	69591	21	41	62	83	104
46	.69466	69340	69214	69088	68962	68835	68709	68582	68455	68327	21	42	63	84	106
47	.68200	68072	67944	67816	67688	67559	67430	67301	67172	67043	21	43	64	86	107
48	.66913	66783	66653	66523	66393	66262	66131	66000	65869	65738	22	44	65	87	109
49	.65606	65474	65342	65210	65077	64945	64812	64679	64546	64412	22	44	66	88	111
50	.64279	64145	64011	63877	63742	63608	63473	63338	63203	63068	22	45	67	90	112
51	.62932	62796	62660	62524	62388	62251	62115	61978	61841	61704	23	45	68	91	114
52	.61566	61429	61291	61153	61015	60876	60738	60599	60460	60321	23	46	69	92	115
53	.60181	60042	59902	59763	59622	59482	59342	59201	59061	58920	23	47	70	94	117
54	.58779	58637	58496	58354	58212	58070	57928	57786	57643	57501	24	47	71	95	118
55	.57358	57215	57071	56928	56784	56641	56497	56353	56208	56064	24	48	72	96	120
56	.55919	55775	55630	55484	55339	55194	55048	54902	54756	54610	24	49	73	97	121
57	.54464	54317	54171	54024	53877	53730	53583	53435	53288	53140	24	49	74	98	123
58	.52992	52844	52696	52547	52399	52250	52101	51952	51803	51653	25	50	74	99	124
59	.51504	51354	51204	51054	50904	50754	50603	50453	50302	50151	25	50	75	100	125
60.	.50000	49849	49697	49546	49394	49242	49090	48938	48786	48634	25	51	76	101	126
61	.48481	48328	48175	48022	47869	47716	47562	47409	47255	47101	26	51	77	102	128
62	.46947	46793	46639	46484	46330	46175	46020	45865	45710	45554	26	52	77	103	129
63	.45399	45243	45088	44932	44776	44620	44464	44307	44151	43994	26	52	78	104	130
64	.43837	43680	43523	43366	43209	43051	42894	42736	42578	42420	26	53	79	105	131
65	.42262	42104	41945	41787	41628	41469	41310	41151	40992	40833	26	53	79	106	132
66	.40674	40514	40355	40195	40035	39875	39715	39555	39394	39234	27	53	80	107	133
67	.39073	38912	38752	38591	38430	38268	38107	37946	37784	37622	27	54	81	107	134
68	.37461	37299	37137	36975	36812	36650	36488	36325	36162	36000	27	54	81	108	135
69	.35837	35674	35511	35347	35184	35021	34857	34694	34530	34366	27	54	82	109	136
70	.34202	34038	33874	33710	33545	33381	33216	33051	32887	32722	27	55	82	110	137
71	.32557	32392	32227	32061	31896	31730	31565	31399	31233	31068	28	55	83	110	138
72	.30902	30736	30570	30403	30237	30071	29904	29737	29571	29404	28	56	83	111	139
73	.29237	29070	28903	28736	28569	28402	28234	28067	27899	27731	28	56	84	112	139
74	.27564	27396	27228	27060	26892	26724	26556	26387	26219	26050	28	56	84	112	140
75	.25882	25713	25545	25376	25207	25038	24869	24700	24531	24362	28	56	85	113	141
76	.24192	24023	23853	23684	23514	23345	23175	23005	22835	22665	28	57	85	113	141
77	.22495	22325	22155	21985	21814	21644	21474	21303	21132	20962	28	57	85	114	142
78	.20791	20620	20450	20279	20108	19937	19766	19595	19423	19252	29	57	86	114	143
79	.19081	18910	18738	18567	18395	18224	18052	17880	17708	17537	29	57	86	114	143
80	.17365	17193	17021	16849	16677	16505	16333	16160	15988	15816	29	57	86	115	144
81	.15643	15471	15299	15126	14954	14781	14608	14436	14263	14090	29	58	86	115	144
82	.13917	13744	13572	13399	13226	13053	12880	12706	12533	12360	29	58	87	115	144
83	.12187	12014	11840	11667	11494	11320	11147	10973	10800	10626	29	58	87	116	145
84	.10453	10279	10106	09932	09758	09585	09411	09237	09063	08889	29	58	87	116	145
85	.08716	08542	08368	08194	08020	07846	07672	07498	07324	07150	29	58	87	116	145
86	.06976	06802	06627	06453	06279	06105	05931	05756	05582	05408	29	58	87	116	145
87	.05234	05059	04885	04711	04536	04362	04188	04013	03839	03664	29	58	87	116	145
88	.03490	03316	03141	02967	02792	02618	02443	02269	02094	01920	29	58	87	117	145
89	.01745	01571	01396	01222	01047	00873	00698	00524	00349	00175	29	58	87	117	145
90	.0000														

NATURAL TANGENTS.

Degrees	0′ 0°.0	6′ 0°.1	12′ 0°.2	18′ 0°.3	24′ 0°.4	30′ 0°.5	36′ 0°.6	42′ 0°.7	48′ 0°.8	54′ 0°.9	Mean Differences. 1	2	3	4	5
0	.00000	00175	00349	00524	00698	00873	01047	01222	01396	01571	29	58	87	116	146
1	.01746	01920	02095	02269	02444	02619	02793	02968	03143	03317	29	58	87	116	146
2	.03492	03667	03842	04016	04191	04366	04541	04716	04891	05096	29	58	87	117	146
3	.05241	05416	05591	05766	05941	06116	06291	06467	06642	06817	29	58	88	117	146
4	06993	07168	07344	07519	07695	07870	08046	08221	08397	08573	29	59	88	117	146
5	.08749	08925	09101	09277	09453	09629	09805	09981	10158	10334	29	59	88	117	147
6	.10510	10687	10863	11040	11217	11394	11570	11747	11924	12101	29	59	88	118	147
7	.12278	12456	12633	12810	12988	13165	13343	13521	13698	13876	30	59	89	118	148
8	.14054	14232	14410	14588	14767	14945	15124	15302	15481	15660	30	59	89	119	149
9	.15838	16017	16196	16376	16555	16734	16914	17093	17273	17453	30	60	90	120	150
10	.17633	17813	17993	18173	18353	18534	18714	18895	19076	19257	30	60	90	120	150
11	.19438	19619	19801	19982	20164	20345	20527	20709	20891	21073	30	60	91	121	152
12	.21256	21438	21621	21804	21986	22169	22353	22536	22719	22903	30	61	92	122	153
13	.23087	23271	23455	23639	23823	24008	24193	24377	24562	24747	31	61	93	124	155
14	.24933	25118	25304	25490	25676	25862	26048	26235	26421	26608	31	62	93	124	155
15	.26795	26982	27169	27357	27545	27732	27920	28109	28297	28486	31	63	94	125	157
16	.28675	28864	29053	29242	29432	29621	29811	30001	30192	30382	32	63	95	127	158
17	.30573	30764	30955	31147	31338	31530	31722	31914	32106	32299	32	64	96	128	160
18	.32492	32685	32878	33072	33266	33460	33654	33848	34043	34238	32	65	97	129	162
19	.34433	34628	34824	35019	35216	35412	35608	35805	36002	36199	33	66	98	131	164
20	.36397	36595	36793	36991	37190	37388	37588	37787	37986	38186	33	66	99	133	166
21	.38386	38587	38787	38988	39190	39391	39593	39795	39997	40200	34	67	101	134	168
22	.40403	40606	40809	41013	41217	41421	41626	41831	42036	42242	34	68	102	136	170
23	.42447	42654	42860	43067	43274	43481	43689	43897	44105	44314	34	69	104	138	173
24	.44523	44732	44942	45152	45362	45573	45784	45995	46206	46418	35	70	105	141	176
25	.46631	46843	47056	47270	47483	47698	47912	48127	48342	48557	36	71	107	143	179
26	.48773	48989	49206	49423	49640	49858	50076	50295	50514	50733	36	73	109	145	182
27	.50953	51173	51393	51614	51835	52057	52279	52501	52724	52947	37	74	111	148	185
28	.53171	53395	53620	53844	54070	54296	54522	54748	54975	55203	38	75	113	151	188
29	.55431	55659	55888	56117	56347	56577	56808	57039	57271	57503	38	77	115	154	192
30	.57735	57968	58201	58435	58670	58905	59140	59376	59612	59849	39	78	118	157	196
31	.60086	60324	60562	60801	61040	61280	61520	61761	62003	62245	40	79	120	160	200
32	.62487	62730	62973	63217	63462	63707	63953	64199	64446	64693	41	82	123	164	205
33	.64941	65189	65438	65688	65938	66189	66440	66692	66944	67197	42	84	126	167	209
34	.67451	67705	67960	68215	68471	68728	68985	69243	69502	69761	43	86	129	171	214
35	.70021	70281	70542	70804	71066	71329	71593	71857	72122	72388	44	88	132	176	219
36	.72654	72921	73189	73457	73726	73996	74267	74538	74810	75082	45	90	135	180	225
37	.75355	75629	75904	76180	76456	76733	77010	77289	77568	77848	46	92	139	185	231
38	.78129	78410	78692	78975	79259	79544	79829	80115	80402	80690	47	95	142	190	237
39	.80978	81268	81558	81849	82141	82434	82727	83022	83317	83613	49	98	147	195	244
40	.83910	84208	84507	84806	85107	85408	85710	86014	86318	86623	50	100	151	201	252
41	.86929	87236	87543	87852	88162	88473	88784	89097	89410	89725	52	103	155	207	259
42	.90040	90357	90674	90993	91313	91633	91955	92277	92601	92926	53	107	160	214	268
43	.93252	93578	93906	94235	94565	94896	95229	95562	95897	96232	55	111	165	221	276
44	.96569	96907	97246	97586	97927	98270	98613	98958	99304	99652	57	114	171	229	286

NATURAL TANGENTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	1.00000	00350	00701	01053	01406	01761	02117	02474	02832	03192	58	118	177	237	296
46	1.03553	03915	04279	04644	05010	05378	05747	06117	06489	06862	61	123	184	245	307
47	1.07237	07613	07990	08369	08749	09131	09514	09899	10285	10672	63	127	191	255	319
48	1.11061	11452	11844	12238	12633	13029	13428	13828	14229	14632	66	132	199	265	331
49	1.15037	15443	15851	16261	16672	17085	17500	17916	18334	18754	69	138	207	276	344
50	1.19175	19599	20024	20451	20879	21310	21742	22176	22612	23050	72	143	216	288	359
51	1.23490	23931	24375	24820	25268	25717	26169	26622	27077	27535	75	150	225	300	375
52	1.27994	28456	28919	29385	29853	30323	30795	31269	31745	32224	78	157	235	314	392
53	1.32704	33187	33673	34160	34650	35142	35637	36134	36633	37134	82	164	247	329	411
54	1.37638	38145	38653	39165	39679	40195	40714	41235	41759	42286	86	172	259	345	431
55	1.42815	43347	43881	44418	44958	45501	46046	46595	47146	47700	91	181	272	362	453
56	1.48256	48816	49378	49944	50512	51084	51658	52235	52816	53400	95	191	286	382	477
57	1.53987	54576	55170	55767	56366	56969	57575	58184	58797	59414	100	201	302	403	504
58	1.60033	60657	61283	61914	62548	63185	63826	64471	65120	65772	106	213	319	426	533
59	1.66428	67088	67752	68419	69091	69766	70466	71129	71817	72509	113	226	339	452	564
60	1.73205	73905	74610	75319	76032	76749	77471	78198	78929	79665	120	240	360	481	600
61	1.80405	81150	81900	82654	83413	84177	84946	85720	86500	87283	128	255	383	511	639
62	1.88073	88867	89667	90472	91282	92098	92920	93746	94579	95417	136	273	409	546	683
63	1.96261	97111	97967	98828	99695	2.00569	2.01449	2.02335	2.03227	2.04125	146	292	438	584	731
64	2.05030	05942	06860	07785	08716	09654	10600	11552	12511	13477	157	314	471	629	786
65	2.14451	15432	16420	17416	18419	19430	20449	21475	22510	23553	169	338	508	677	846
66	2.24604	25663	26730	27806	28891	29984	31086	32197	33317	34447	183	366	549	732	915
67	2.35585	36733	37891	39058	40235	41421	42618	43825	45043	46270	199	397	596	795	994
68	2.47509	48758	50018	51289	52571	53865	55170	56487	57815	59156	Mean differences cease to be sufficiently accurate.				
69	2.60509	61874	63252	64642	66046	67462	68892	70335	71792	73263					
70	2.74748	76247	77761	79289	80833	82391	83965	85556	87161	88783					
71	2.90421	92076	93748	95437	97144	98868	3.00611	3.02372	3.04152	3.05950					
72	3.07768	09606	11464	13341	15240	17159	19100	21063	23048	25055					
78	3.27085	29139	31216	33317	35443	37594	39771	41973	44202	46458					
74	3.48741	51053	53393	55761	58160	60588	63048	65538	68061	70616					
75	3.73205	75828	78485	81177	83906	86671	89474	92316	95196	98117					
76	4.01078	04081	07127	10216	13350	16530	19756	23030	26352	29724					
77	4.33148	36623	40152	43735	47374	51071	54826	58641	62518	66458					
78	4.70463	74534	78673	82882	87162	91516	95945	5.00451	5.05037	5.09704					
79	5.14455	19293	24218	29235	34345	39552	44857	50264	55777	61397					
80	5.67128	72974	78938	85024	91236	97576	6.04051	6.10664	6.17419	6.24321					
81	6.31375	38587	45961	53503	61220	69116	77199	85475	93952	7.02637					
82	7.11537	20661	30018	39616	49465	59575	69957	80622	91582	8.02848					
83	8.14435	26356	38625	51259	64275	77689	91520	9.05789	9.20516	9.35724					
84	9.51436	9.6768	9.8448	10.019	10.199	10.385	10.579	10.780	10.988	11.205					
85	11.430	11.664	11.909	12.163	12.429	12.706	12.996	13.300	13.617	13.951					
86	14.301	14.669	15.056	15.464	15.895	16.350	16.832	17.343	17.886	18.464					
87	19.081	19.740	20.446	21.205	22.022	22.904	23.859	24.898	26.031	27.271					
88	28.636	30.145	31.821	33.694	35.801	38.188	40.917	44.066	47.740	52.081					
89	57.290	63.657	71.615	81.847	95.489	114.60	143.24	190.98	286.48	572.96					
90	00														

NATURAL COSECANTS.

Numbers in difference columns to be subtracted, not added.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	Inf.	572.96	286.48	190.99	143.24	114.59	95.495	81.853	71.622	63.665					
1	57.299	52.090	47.750	44.077	40.930	38.202	35.815	33.708	31.836	30.161					
2	28.654	27.290	26.050	24.918	23.880	22.926	22.044	21.229	20.471	19.766					
3	19.107	18.492	17.914	17.372	16.862	16.380	15.926	15.496	15.089	14.703					
4	14.336	13.987	13.654	13.337	13.035	12.745	12.469	12.204	11.951	11.707					
5	11.474	11.249	11.034	10.826	10.626	10.433	10.248	10.068	9.8955	9.7283					
6	9.5668	9.4105	9.2593	9.1129	8.9711	8.8337	8.7004	8.5711	8.4457	8.3238					
7	8.2055	8.0905	7.9787	7.8700	7.7642	7.6613	7.5611	7.4635	7.3684	7.2757					
8	7.1853	7.0972	7.0112	6.9273	6.8454	6.7655	6.6874	6.6111	6.5366	6.4637					
9	6.3925	6.3228	6.2546	6.1880	6.1227	6.0589	5.9963	5.9351	5.8751	5.8164					
10	5.7588	5.7023	5.6470	5.5928	5.5396	5.4874	5.4362	5.3860	5.3367	5.2883					
11	5.2408	5.1942	5.1484	5.1034	5.0593	5.0159	4.9732	4.9313	4.8901	4.8496					
12	4.8097	4.7706	4.7321	4.6942	4.6569	4.6202	4.5841	4.5486	4.5137	4.4793					
13	4.4454	4.4121	4.3792	4.3469	4.3150	4.2837	4.2527	4.2223	4.1923	4.1627					
14	4.1336	4.1048	4.0765	4.0486	4.0211	3.9939	3.9672	3.9408	3.9147	3.8890					
15	3.86370	83871	81404	78970	76568	74198	71858	69549	67269	65018					
16	3.62796	60601	58434	56294	54181	52094	50032	47995	45983	43995					
17	3.42030	40089	38171	36276	34403	32551	30721	28912	27123	25355					
18	3.23607	21878	20169	18479	16808	15155	13520	11903	10303	08721	274	548	822	1047	1371
19	3.07155	05607	04075	02559	01059	2.99574	2.98106	2.96652	2.95213	2.93790	246	492	738	985	1231
20	2.92380	90986	89605	88238	86885	85545	84219	82906	81605	80318	222	445	667	889	1111
21	2.79043	77780	76530	75292	74065	72850	71647	70455	69275	68105	201	403	604	806	1008
22	2.66947	65799	64662	63535	62419	61313	60217	59130	58054	56988	181	367	550	734	918
23	2.55930	54883	53845	52815	51795	50784	49782	48789	47804	46827	168	335	504	671	839
24	2.45859	44900	43948	43005	42070	41142	40222	39311	38406	37509	154	308	462	616	770
25	2.36620	35738	34863	33996	33135	32282	31436	30596	29763	28937	142	283	425	566	708
26	2.28117	27304	26498	25697	24903	24116	23334	22559	21790	21026	130	262	392	523	654
27	2.20269	19517	18771	18031	17297	16568	15845	15127	14414	13707	121	242	363	484	606
28	2.13005	12309	11617	10931	10250	09574	08903	08236	07575	06918	112	225	337	449	561
29	2.06267	05619	04977	04339	03706	03077	02453	01833	01218	00607	104	209	313	417	522
30	2.00000	1.99398	1.98799	1.98205	1.97615	1.97029	1.96448	1.95870	1.95296	1.94726	97	194	292	389	486
31	1.94160	93598	93040	92486	91935	91388	90845	90305	89769	89237	90	182	272	363	454
32	1.88708	88183	87661	87142	86627	86116	85608	85103	84601	84103	85	170	255	340	425
33	1.83608	83116	82627	82142	81659	81180	80704	80231	79761	79293	79	159	239	318	398
34	1.78829	78368	77910	77454	77001	76552	76105	75661	75219	74781	75	149	224	299	374
35	1.74345	73911	73481	73053	72628	72205	71785	71367	70953	70540	70	140	211	281	351
36	1.70130	69723	69318	68915	68515	68117	67722	67329	66938	66550	66	132	198	264	331
37	1.66164	65780	65399	65020	64643	64268	63895	63525	63157	62791	62	124	187	249	311
38	1.62427	62065	61705	61348	60992	60639	60287	59938	59590	59245	59	117	176	235	294
39	1.58902	58560	58221	57883	57547	57213	56881	56551	56223	55897	55	111	166	221	278
40	1.55572	55250	54929	54610	54292	53977	53663	53351	53041	52732	52	105	157	210	262
41	1.52425	52120	51817	51515	51215	50916	50619	50324	50030	49738	49	99	149	198	248
42	1.49448	49159	48871	48586	48301	48019	47738	47458	47180	46903	47	94	141	188	235
43	1.46628	46354	46082	45811	45542	45274	45007	44742	44479	44217	44	89	133	178	223
44	1.43956	43696	43438	43181	42926	42672	42419	42168	41918	41669	42	84	126	169	211

NATURAL COSECANTS.

Numbers in difference columns to be subtracted, not added.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	1.41421	41175	40930	40687	40444	40203	39963	39725	39487	39251	40	79	120	160	200
46	1.39016	38783	38550	38319	38089	37860	37632	37406	37180	36956	38	76	114	152	190
47	1.36733	36511	36290	36070	35852	35634	35418	35203	34988	34775	36	71	108	145	181
48	1.34563	34352	34142	33934	33726	33519	33314	33109	32905	32703	34	69	103	137	172
49	1.32501	32301	32101	31903	31705	31509	31313	31119	30925	30732	32	65	98	130	163
50	1.30541	30350	30160	29971	29784	29597	29411	29226	29041	28858	31	62	93	124	155
51	1.28676	28495	28314	28134	27956	27778	27601	27425	27250	27075	30	59	89	118	148
52	1.26902	26729	26557	26387	26216	26047	25879	25711	25545	25379	28	56	84	113	141
53	1.25214	25049	24885	24723	24561	24400	24240	24081	23922	23764	27	54	80	107	134
54	1.23607	23450	23295	23140	22986	22833	22680	22528	22377	22227	26	51	77	102	128
55	1.22077	21929	21781	21633	21487	21341	21195	21051	20907	20764	24	49	73	97	121
56	1.20622	20480	20339	20199	20059	19920	19782	19645	19508	19372	23	46	69	92	116
57	1.19236	19102	18967	18834	18701	18569	18437	18307	18176	18047	22	44	66	88	110
58	1.17918	17790	17662	17535	17409	17283	17158	17033	16909	16786	21	42	63	84	105
59	1.16663	16541	16420	16299	16179	16059	15940	15822	15704	15587	20	40	60	80	99
60	1.15470	15354	15239	15124	15009	14896	14782	14670	14558	14446	19	38	57	76	95
61	1.14335	14225	14115	14006	13897	13789	13682	13575	13468	13362	18	36	54	72	90
62	1.13257	13152	13048	12944	12841	12738	12636	12534	12433	12333	17	34	51	68	85
63	1.12233	12133	12034	11936	11838	11740	11643	11547	11451	11355	16	32	49	65	81
64	1.11260	11166	11072	10978	10885	10793	10701	10609	10518	10428	15	31	46	61	77
65	1.10338	10248	10159	10071	09982	09895	09808	09721	09635	09549	15	29	44	58	73
66	1.09464	09379	09294	09211	09127	09044	08962	08880	08798	08717	14	28	41	55	69
67	1.08636	08556	08476	08397	08318	08239	08161	08084	08006	07930	13	26	39	52	64
68	1.07853	07778	07702	07627	07553	07479	07405	07332	07259	07186	12	25	37	49	62
69	1.07115	07043	06972	06901	06831	06761	06691	06622	06554	06486	12	23	35	46	58
70	1.06418	06350	06283	06217	06151	06085	06020	05955	05890	05826	11	22	33	44	55
71	1.05762	05699	05636	05573	05511	05449	05388	05327	05266	05206	10	21	31	41	51
72	1.05146	05087	05028	04969	04911	04853	04795	04738	04682	04625	10	19	29	38	48
73	1.04569	04514	04458	04403	04349	04295	04241	04188	04135	04082	9	18	26	34	44
74	1.04030	03978	03927	03875	03825	03774	03724	03674	03625	03576	8	17	25	33	42
75	1.03528	03479	03432	03384	03337	03290	03244	03197	03152	03106	8	16	23	31	39
76	1.03061	03017	02972	02928	02885	02842	02799	02756	02714	02672	7	14	22	29	36
77	1.02630	02589	02548	02508	02468	02428	02388	02349	02311	02272	7	13	20	26	33
78	102234	02196	02159	02122	02085	02049	02013	01977	01941	01906	6	12	18	24	30
79	1.01872	01837	01803	01769	01736	01703	01670	01638	01606	01574	5	11	16	22	27
80	1.01543	01512	01481	01450	01420	01391	01361	01332	01303	01275	5	10	15	20	24
81	1.01247	01219	01191	01164	01137	01111	01084	01059	01033	01008	4	9	13	18	22
82	1.00983	00958	00934	00910	00886	00863	00840	00817	00795	00773	4	8	12	15	19
83	1.00751	00730	00708	00688	00667	00647	00627	00608	00588	00569	3	7	10	13	17
84	1.00551	00533	00515	00497	00480	00463	00446	00429	00413	00397	3	6	8	11	14
85	1.00382	00367	00352	00337	00323	00309	00296	00283	00269	00257	2	5	7	9	12
86	1.00244	00232	00220	00209	00198	00187	00176	00166	00156	00147	2	4	5	7	9
87	1.00137	00128	00120	00111	00103	00095	00088	00081	00074	00067	1	3	4	5	6
88	1.00061	00055	00049	00044	00039	00034	00030	00026	00022	00018	1	2	2	3	4
89	1.00015	00012	00010	00007	00005	00004	00002	00001	00001	00000	0	1	1	1	1

NATURAL COSECANTS.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	1.00000	00000	00001	00001	00002	00004	00005	00007	00010	00012	0	1	1	1	1
1	1.00015	00018	00022	00026	00030	00034	00039	00044	00049	00055	1	2	2	3	4
2	1.00061	00067	00074	00081	00088	00095	00103	00111	00120	00128	1	3	4	5	6
3	1.00137	00147	00156	00166	00176	00187	00198	00209	00220	00232	2	4	5	7	9
4	1.00244	00257	00269	00282	00296	00309	00323	00337	00352	00367	2	5	7	9	12
5	1.00382	00397	00413	00429	00446	00463	00480	00497	00515	00533	3	6	8	11	14
6	1.00551	00569	00588	00608	00627	00647	00667	00688	00708	00730	3	7	10	13	17
7	1.00751	00773	00795	00817	00840	00863	00886	00910	00934	00958	4	8	12	15	19
8	1.00983	01008	01033	01059	01084	01111	01137	001164	001191	01219	4	9	13	18	22
9	1.01247	01275	01303	01332	01361	01391	01420	01450	01481	01512	5	10	15	20	24
10	1.01543	01574	01606	01638	01670	01703	01736	01769	01803	01837	5	11	16	22	27
11	1.01872	01906	01941	01977	02013	02049	02085	02122	02159	02196	6	12	18	24	30
12	1.02234	02272	02311	02349	02388	02428	02468	02508	02548	02589	7	13	20	26	33
13	1.02630	02672	02714	02756	02799	02842	02885	02928	02972	03017	7	14	22	29	36
14	1.03061	03106	03152	03197	03244	03290	03337	03384	03432	03479	8	16	23	31	39
15	1.03528	03576	03625	03674	03724	03774	03825	03875	03927	03978	8	17	25	33	42
16	1.04030	04082	04135	04188	04241	04295	04349	04403	04458	04514	9	18	26	34	44
17	1.04569	04625	04682	04738	04795	04853	04911	04969	05028	05087	10	19	29	38	48
18	1.05146	05206	05266	05327	05388	05449	05511	05573	05636	05699	10	21	31	41	51
19	1.05762	05826	05890	05955	06020	06085	06151	06217	06283	06350	11	22	33	44	55
20	1.06418	06486	06554	06622	06691	06761	06831	06901	06972	07043	12	23	35	46	58
21	1.07114	07186	07259	07332	07405	07479	07553	07627	07702	07778	12	25	37	49	62
22	1.07853	07930	08006	08084	08161	08239	08318	08397	08476	08556	13	26	39	52	64
23	1.08636	08717	08798	08880	08962	09044	09127	09211	09294	09379	14	28	41	55	69
24	1.09464	09549	09635	09721	09808	09895	09982	10071	10159	10248	15	29	44	58	73
25	1.10338	10428	10518	10609	10701	10793	10885	10978	11072	11166	15	31	46	61	77
26	1.11260	11355	11451	11547	11643	11740	11838	11936	12034	12133	16	32	49	65	81
27	1.12233	12333	12433	12534	12636	12738	12841	12944	13048	13152	17	34	51	68	85
28	1.13257	13362	13468	13575	13682	13789	13897	14006	14115	14225	18	36	54	72	90
29	1.14335	14446	14558	14670	14782	14896	15009	15124	15239	15354	19	38	57	76	95
30	1.15470	15587	15704	15822	15940	16059	16179	16299	16420	16541	20	40	60	80	99
31	1.16663	16786	16909	17033	17158	17283	17409	17535	17662	17790	21	42	63	84	105
32	1.17918	18047	18176	18307	18437	18569	18701	18834	18967	19102	22	44	66	88	110
33	1.19236	19372	19508	19645	19782	19920	20059	20199	20339	20480	23	46	69	92	116
34	1.20622	20764	20907	21051	21195	21341	21487	21633	21781	21929	24	49	73	97	121
35	1.22077	22227	22377	22528	22680	22833	22986	23140	23295	23450	26	51	77	102	128
36	1.23607	23764	23922	24081	24240	24400	24561	24723	24886	25049	27	54	80	107	134
37	1.25214	25379	25545	25711	25879	26047	26216	26387	26557	26729	28	56	84	113	141
38	1.26902	27075	27250	27425	27601	27778	27956	28134	28314	28495	30	59	89	118	148
39	1.28676	28858	29042	29226	29411	29597	29784	29971	30100	30350	31	62	93	124	155
40	1.30541	30732	30925	31119	31313	31509	31705	31903	32101	32301	32	65	98	130	163
41	1.32501	32703	32905	33109	33314	33519	33726	33934	34142	34352	34	69	103	137	172
42	1.34563	34775	34988	35203	35418	35634	35852	36070	36290	36511	36	71	108	145	181
43	1.36733	36956	37180	37406	37632	37860	38089	38319	38550	38783	38	76	114	152	190
44	1.39016	39251	39487	39725	39963	40203	40444	40687	40930	41175	40	80	120	160	200

NATURAL COSECANTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	1.41421	41669	41918	42168	42419	42672	42926	43181	43438	43696	42	84	126	169	211
46	1.43956	44217	44479	44742	45007	45274	45542	45811	46082	46354	44	89	133	178	223
47	1.46628	46903	47180	47458	47738	48019	48301	48586	48871	49159	47	94	141	188	235
48	1.49448	49738	50030	50342	50619	50916	51215	51515	51817	52120	49	99	149	198	248
49	1.52425	52732	53041	53351	53663	53977	54292	54610	54929	55250	52	105	157	209	262
50	1.55572	55897	56223	56551	56881	57213	57547	57883	58221	58560	55	111	166	222	278
51	1.58902	59245	59590	59938	60287	60639	60992	61348	61705	62065	59	117	176	235	294
52	1.62427	62791	63157	63525	63895	64268	64643	65020	65399	65780	62	124	187	249	311
53	1.66164	66550	66938	67329	67722	68117	68515	68915	69318	69723	66	132	198	264	331
54	1.70130	70540	70953	71367	71785	72205	72628	73053	73481	73911	70	140	211	281	351
55	1.74345	74781	75219	75661	16705	76552	77001	77454	77910	78368	75	149	224	299	374
56	1.78829	79293	79761	80231	80704	81180	81659	82142	82627	83116	79	159	239	318	398
57	1.83608	84103	84601	85103	85608	86116	86627	87142	87661	88183	85	170	253	340	425
58	1.88708	89237	89769	90305	90845	91388	91935	92486	93040	93598	90	182	272	363	454
59	1.94160	94726	95296	95870	96448	97029	97615	98205	98799	99398	97	194	292	389	486
60	2.0000	00607	01218	01833	02453	03077	03706	04339	04977	05619	104	209	313	417	522
61	2.06267	06918	07575	08236	08903	09574	10250	10931	11617	12309	112	225	339	449	561
62	2.13005	13707	14414	15127	15845	16568	17297	18031	18772	19517	121	242	363	484	606
63	2.20269	21026	21790	22559	23334	24116	24903	25697	26498	27304	130	262	392	523	654
64	2.28117	28937	29763	30596	31436	32282	33135	33996	34863	35738	142	283	425	567	709
65	2.36620	37509	38406	39311	40222	41142	42070	43005	43948	44900	154	308	462	616	770
66	2.45859	46827	47804	48789	49782	50784	51795	52815	53845	54883	168	335	504	672	839
67	2.55930	56988	58054	59130	60217	61313	62419	63535	64662	65799	183	367	550	734	918
68	2.66947	68105	69275	70455	71647	72850	74065	75292	76530	77780	200	403	605	806	1008
69	2.79043	80318	81605	82906	84219	85545	56885	88238	89605	90986	Mean differences cease				
70	2.92380	93790	95213	96652	98106	99574	3.01059	3.02559	3.04075	3.05607	to be sufficiently accurate.				
71	3.07155	08721	10303	11903	13520	15155	16808	18479	20169	21878					
72	3.23607	25355	27123	28912	30721	32551	34403	36276	38171	40089					
73	3.42030	43995	45983	47995	50032	52094	54181	56294	58434	60601					
74	3.62796	65018	67269	69549	71858	74198	76568	78970	81404	83871					
75	3.86370	88904	91473	94076	96716	99393	4.02107	4.04860	4.07652	4.10484					
76	4.13357	16271	19228	22229	25275	28366	31503	34689	37923	41206					
77	4.44541	47928	51368	54863	58414	62023	65690	69417	73205	77057					
78	4.80973	84956	89007	93128	97320	5.01585	5.05926	5.10344	5.14842	5.19421					
79	5.24084	28833	33671	38600	43622	48740	53958	59277	64701	70234					
80	5.75877	81635	87511	93509	99633	6.05886	6.12273	6018797	6025464	6.32279					
81	6.39245	46369	53655	61110	68738	76547	84542	92731	7.01120	7.09717					
82	7.18530	27566	36835	46346	56107	66130	76424	87001	97873	8.09052					
83	8.20551	32384	44566	57113	70041	83367	97111	9.11292	9.25931	9.41052					
84	9.56677	9.7283	9.8955	10.068	10.248	10.433	10.626	10.826	11.034	11.249					
85	11.474	11.707	11.951	12.204	12.469	12.745	13.035	13.337	13.654	13.987					
86	14.336	14.703	15.089	15.496	15.926	16.380	16.862	17.372	17.914	18.492					
87	19.107	19.766	20.471	21.229	22.044	22.926	23.880	24.918	26.050	27.290					
88	28.654	30.161	31.836	33.708	35.815	38.202	40.930	44.077	47.750	52.090					
89	57.299	63.665	71.622	81.853	95.495	114.59	143.24	190.99	286.48	572.96					

NATURAL COTANGENTS.

Numbers in difference columns to be subtracted, not added.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	00	572.96	286.48	190.98	143.24	114.59	95.489	81.847	71.615	63.657					
1	57.290	52.081	47.740	44.066	40.917	38.188	35.801	33.694	31.821	30.145					
2	28.636	27.271	26.031	24.898	23.859	22.904	22.022	21.205	20.446	19.740					
3	19.081	18.464	17.886	17.343	16.832	16.350	15.895	15.464	15.056	14.669					
4	14.301	13.951	13.617	13.300	12.996	12.706	12.429	12.163	11.909	11.664					
5	11.430	11.205	10.998	10.780	10.579	10.385	10.199	10.019	9.8448	9.6768					
6	9.5144	9.3572	9.2052	9.0579	8.9152	8.7769	8.6427	8.5126	8.3863	8.2636					
7	8.1443	8.0285	7.9158	7.8062	7.6996	7.5958	7.4947	7.3962	7.3002	7.2066					
8	7.1154	7.0264	6.9395	6.8548	6.7720	6.6912	6.6122	6.5350	6.4596	6.3859					
9	6.3138	6.2432	6.1742	6.1066	6.0405	5.9758	5.9124	5.8502	5.7894	5.7297					
10	5.67128	61397	55777	50264	44857	39552	34345	29235	24218	19293					
11	5.14455	09704	05037	00451	4.95945	4.91516	4.87162	4.82882	4.78673	4.74534					
12	4.70463	66458	62518	58641	54826	51071	47374	43735	40152	36623					
13	4.33148	29724	26352	23030	19756	16530	13350	10216	07127	04081					
14	4.01078	3.98117	3.95196	3.92316	3.89474	3.86671	3.83906	3.81177	3.78485	3.75828					
15	3.73205	70616	68061	65538	63048	60588	58160	55761	53393	51053					
16	3.48741	46458	44202	41973	39771	37594	35443	33317	31216	29139					
17	3.27085	25055	23048	21063	19100	17159	15240	13341	11464	09606					
18	3.07768	05950	04152	02372	00611	2.98868	2.97144	2.95437	2.93748	2.92076					
19	2.90421	88783	87161	85555	83965	82391	80833	79289	77761	76247					
20	2.74748	73263	71792	70335	68892	67462	66045	64642	63252	61874					
21	2.60509	59156	57815	56487	55170	53865	52571	51289	50018	48758					
22	2.47509	46270	45043	43825	42618	41421	40235	39058	37891	36733	199	397	596	795	994
23	2.35585	34447	33317	32197	31086	29984	28891	27806	26730	25663	183	366	549	732	916
24	2.24604	23553	22510	21475	20449	19430	18419	17416	16420	15432	169	338	508	677	846
25	2.14451	13477	12511	11552	10600	09654	08716	07785	06860	05942	157	314	471	629	786
26	2.05030	04125	03227	02335	01449	00569	1.99695	1.98828	1.97966	1.97111	146	292	438	584	731
27	1.96261	95417	94579	93746	92920	92098	91282	90472	89667	88867	136	273	409	546	683
28	1.88073	87283	86499	85720	84946	84177	83413	82654	81899	81150	128	255	383	511	639
29	1.80405	79665	78929	78198	77471	76749	76032	75319	74610	73905	120	240	360	481	600
30	1.73205	72509	71817	71129	70446	69766	69091	68419	67752	67088	113	226	339	452	564
31	1.66428	65772	65120	64471	63826	63185	62548	61914	61283	60657	106	213	319	428	533
32	1.60033	59414	58797	58184	57575	56969	56366	55766	55170	54576	101	201	302	403	504
33	1.53986	53400	52816	52235	51658	51084	50512	49944	49378	48816	95	191	286	382	477
34	1.48256	47699	47146	46595	46046	45501	44958	44418	43881	43347	91	181	272	362	453
35	1.42815	42286	41759	41235	40714	40195	39679	39165	38653	38145	86	172	259	345	431
36	1.37638	37134	36633	36134	35637	35142	34650	34160	33673	33187	82	164	247	329	411
37	1.32704	32224	31745	31269	30795	30323	29853	29385	28919	28456	78	157	235	314	392
38	1.27994	27535	27077	26622	26169	25717	25268	24820	24375	23931	75	150	225	300	375
39	1.23490	23050	22612	22176	21742	21310	20879	20451	20024	19599	72	143	216	288	359
40	1.19175	18754	18334	17916	17500	17085	16672	16261	15851	15443	69	138	207	276	343
41	1.15037	14632	14229	13828	13428	13029	12633	12238	11844	11452	66	132	199	265	331
42	1.11061	10672	10285	09899	09514	09131	08749	08369	07990	07613	63	127	191	255	319
43	1.07237	06862	06489	06117	05747	05378	05010	04644	04279	03915	61	123	184	245	307
44	1.03553	03192	02832	02474	02117	01761	01406	01053	00701	00350	58	118	177	237	296

NATURAL COTANGENTS.

Numbers in difference columns to be subtracted, not added

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	1.0000	99652	99304	98958	98613	98270	97927	97586	97246	96907	57	114	171	229	286
46	0.96569	96232	95897	95562	95229	94896	94565	94235	93906	93578	55	111	165	221	276
47	0.93252	92926	92601	92277	91955	91633	91313	90993	90674	90357	53	107	160	214	268
48	0.90040	89725	89410	89097	88784	88473	88162	87852	87543	87236	52	103	155	207	259
49	0.86929	86623	86318	86014	85710	85408	85107	84806	84507	84208	50	100	151	201	252
50	0.83910	83613	83317	83022	82727	82434	82141	81849	81558	81268	49	98	147	195	244
51	0.80978	80690	80402	80115	79829	79544	79259	78975	78692	78410	47	95	142	190	237
52	0.78129	77848	77568	77289	77010	76733	76755	76180	75904	75629	46	92	139	185	231
53	0.75355	75082	74810	74538	74267	73996	73726	73457	73189	72921	45	90	135	180	225
54	0.72654	72388	72122	71857	71593	71329	71066	70804	70542	70281	44	88	132	176	219
55	0.70021	69761	69502	69243	68985	68728	68471	68215	67960	67705	43	86	129	171	214
56	0.67451	67197	66944	66692	66440	66189	65938	65688	65438	65189	42	84	126	167	209
57	0.64941	64693	64446	64199	63953	63707	63462	63217	62973	62730	41	82	123	164	205
58	0.62487	62245	62003	61761	61520	61280	61040	60801	60562	60324	40	79	120	160	200
59	0.60086	59849	59612	59376	59140	58905	58670	58435	58201	57968	39	78	118	157	196
60	0.57735	57503	57271	57039	56808	56577	56347	56117	55888	55659	38	77	115	154	192
61	0.55431	55203	54975	54748	54522	54296	54070	53844	53620	53395	38	75	113	151	188
62	0.53171	52947	52724	52501	52279	52057	51835	51614	51393	51173	37	74	111	148	185
63	0.50953	50733	50514	50295	50076	49858	49640	49423	49206	48989	36	73	109	145	182
64	0.48773	48557	48342	48127	47912	47698	47483	47270	47056	46843	36	71	107	143	179
65	0.46631	46418	46206	45995	45784	45573	45362	45152	44942	44732	35	70	105	141	176
66	0.44523	44314	44105	43897	43689	43481	43274	43067	42860	42654	35	69	104	138	173
67	0.42447	42242	42036	41831	41626	41421	41217	41013	40809	40606	34	68	102	136	170
68	0.40403	40200	39997	39795	39593	39391	39190	38988	38787	38587	34	67	101	134	168
69	0.38386	38186	37986	37787	37588	37388	37190	36991	36793	36595	33	66	99	133	166
70	0.36397	36199	36002	35805	35608	35412	35216	35019	34824	34628	33	66	98	131	164
71	0.34433	34238	34043	33848	33654	33460	33266	33072	32878	32685	32	65	97	129	162
72	0.32492	32299	32106	31914	31722	31530	31338	31147	30955	30764	32	64	96	128	160
73	0.30573	30382	30192	30001	29811	29621	29432	29242	29053	28864	32	63	95	127	158
74	0.28675	28486	28297	28109	27920	27732	27545	27357	27169	26982	31	63	94	125	157
75	0.26795	26608	26421	26235	26048	25862	25676	25490	25304	25118	31	62	93	124	155
76	0.24933	24747	24562	24377	24193	24008	23823	23639	23455	23271	31	61	93	124	155
77	0.23087	22903	22719	22536	22353	22169	21986	21804	21621	21438	31	61	92	122	153
78	0.21256	21073	20891	20709	20527	20345	20164	19982	19801	19619	30	60	91	121	152
79	0.19438	19257	19076	18895	18714	18534	18353	18173	17993	17813	30	60	90	120	150
80	0.17633	17453	17273	17093	16914	16734	16555	16376	16196	16017	30	60	90	120	150
81	0.15838	15660	15481	15302	15123	14945	14767	14588	14410	14232	30	59	89	119	149
82	0.14054	13876	13698	13521	13343	13165	12988	12810	12633	12456	30	59	89	118	148
83	0.12278	12101	11924	11747	11570	11394	11217	11040	10863	10687	29	59	88	118	147
84	0.10510	10334	10158	09981	09805	09629	09453	09277	09101	08925	29	59	88	117	147
85	0.08749	08573	08397	08221	08046	07870	07695	07519	07344	07168	29	59	88	117	146
86	0.06993	06817	06642	06467	06291	06116	05941	05766	05591	05416	29	58	88	117	146
87	0.05241	05066	04891	04716	04541	04366	04191	04016	03842	03667	29	58	87	117	146
88	0.03492	03317	03143	02968	02793	02619	02444	02269	02095	01920	29	58	87	116	146
89	0.01746	01571	01396	01222	01047	00873	00698	00524	00349	00175	29	58	87	116	146

LOGARITHMS OF SINES.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	∞	$\bar{3}$.24188	$\bar{3}$.54291	71900	84393	94084	$\bar{2}$.02002	$\bar{2}$.08696	$\bar{2}$.14495	$\bar{2}$.19610					
1	$\bar{2}$.24186	28324	32103	35578	38796	41792	44594	47226	49708	52055					
2	$\bar{2}$.54282	56400	58419	60349	62196	63968	65670	67308	68886	70409					
3	$\bar{2}$.71880	73303	74680	76015	77310	78568	79790	80978	82134	83261					
4	$\bar{2}$.84358	85429	86474	87494	88490	89464	90417	91349	92261	93154					
5	$\bar{2}$.94030	94887	95728	96553	97363	98157	98937	99704	$\bar{1}$.00456	$\bar{1}$.01196	132	263	395	526	658
6	$\bar{1}$.01923	02639	03342	04034	04715	05386	06046	06696	07337	07968	111	222	333	444	555
7	$\bar{1}$.08589	09202	09807	10402	10990	11570	12142	12706	13263	13813	96	192	288	384	480
8	$\bar{1}$.14536	14891	15421	15944	16460	16970	17474	17973	18465	18952	84	169	253	338	423
9	$\bar{1}$.19433	19909	20380	20845	21306	21761	22211	22657	23098	23535	76	151	227	302	378
10	$\bar{1}$.23967	24395	24818	25237	25652	26063	26470	26873	27273	27668	68	136	205	273	341
11	$\bar{1}$.28060	28448	28833	29214	29591	29966	30336	30704	31068	31430	62	124	186	248	311
12	$\bar{1}$.31788	32143	32495	32844	33190	33534	33874	34212	34547	34879	57	114	171	228	285
13	$\bar{1}$.35209	35536	35860	36182	36502	36819	37133	37445	37755	38062	53	105	158	211	263
14	$\bar{1}$.38368	38670	38971	39270	39566	39860	40152	40442	40730	41016	49	98	146	195	244
15	$\bar{1}$.41300	41582	41861	42139	42416	42690	42962	43233	43502	43769	45	91	136	182	228
16	$\bar{1}$.44034	44297	44559	44819	45077	45334	45589	45843	46095	46345	42	85	128	170	213
17	$\bar{1}$.46594	46841	47086	47330	47573	47814	48054	48292	48529	48764	40	80	120	160	200
18	$\bar{1}$.48998	49231	49462	49692	49920	50148	50374	50598	50821	51043	37	75	113	151	189
19	$\bar{1}$.51264	51484	51702	51919	52135	52350	52563	52775	52986	53196	35	71	107	143	178
20	$\bar{1}$.53405	53613	53819	54025	54229	54433	54635	54836	55036	55235	34	68	101	135	169
21	$\bar{1}$.55433	55630	55826	56021	56215	56408	56599	56790	56980	57169	33	64	96	128	160
22	$\bar{1}$.57358	57545	57731	57916	58101	58284	58467	58648	58829	59009	31	61	91	122	152
23	$\bar{1}$.59188	59366	59543	59720	59895	60070	60243	60417	60589	60761	29	58	87	116	145
24	$\bar{1}$.60931	61101	61270	61438	61606	61773	61939	62104	62268	62432	28	56	83	111	138
25	$\bar{1}$.62595	62757	62918	63079	63239	63398	63557	63715	63872	64028	27	53	80	106	132
26	$\bar{1}$.64184	64339	64494	64647	64800	64953	65104	65255	65406	65556	26	51	77	101	126
27	$\bar{1}$.65705	65853	66001	66148	66295	66441	66586	66731	66875	67018	25	49	73	97	121
28	$\bar{1}$.67161	67303	67445	67586	67726	67866	68006	68144	68283	68420	24	47	70	93	116
29	$\bar{1}$.68557	68694	68829	68965	69100	69234	69368	69501	69633	69765	23	45	67	89	112
30	$\bar{1}$.69897	70028	70159	70288	70418	70547	70675	70803	70931	71058	21	43	64	86	107
31	$\bar{1}$.71184	71310	71435	71560	71685	71809	71932	72055	72177	72299	21	41	62	82	103
32	$\bar{1}$.72421	72542	72663	72783	72902	73022	73140	73259	73377	73494	20	40	60	79	99
33	$\bar{1}$.73611	73727	73843	73959	74074	74189	74303	74417	74531	74644	19	38	57	76	95
34	$\bar{1}$.74756	74868	74980	75091	75202	75313	75423	75533	75642	75751	18	37	55	74	92
35	$\bar{1}$.75859	75967	76075	76182	76289	76395	76501	76607	76712	76817	18	35	53	71	89
36	$\bar{1}$.76922	77026	77130	77233	77336	77439	77541	77643	77744	77846	17	34	51	68	85
37	$\bar{1}$.77946	78047	78147	78246	78346	78445	78543	78642	78739	78837	16	33	49	66	82
38	$\bar{1}$.78934	79031	79128	79224	79319	79415	79510	79605	79699	79793	16	32	48	64	79
39	$\bar{1}$.79887	79981	80074	80166	80259	80351	80443	80534	80625	80716	15	31	46	61	77
40	$\bar{1}$.80807	80897	80987	81076	81166	81254	81343	81431	81519	81607	15	30	44	59	74
41	$\bar{1}$.81694	81781	81868	81954	82041	82126	82212	82297	82382	82467	14	29	43	57	71
42	$\bar{1}$.82551	82635	82719	82802	82885	82968	83051	83133	83215	83297	14	28	41	55	69
43	$\bar{1}$.83378	83459	83540	83621	83701	83781	83861	83940	84020	84098	13	27	40	53	67
44	$\bar{1}$.84177	84255	84336	84411	84489	84566	84643	84720	84796	84873	13	26	39	51	64

LOGARITHMS OF SINES

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	$\bar{1}$.84948	85024	85100	85175	85250	85324	85399	85473	85546	85620	12	25	37	50	62
46	$\bar{1}$.85693	85766	85839	85912	85984	86056	86128	86200	86271	86342	12	24	36	48	60
47	$\bar{1}$.86413	86483	86554	86624	86694	86763	86832	86902	86970	87039	12	23	36	46	58
48	$\bar{1}$.87107	87175	87243	87311	87378	87446	87513	87579	87646	87712	11	22	34	45	56
49	$\bar{1}$.87778	87844	87909	87975	88040	88105	88169	88234	88298	88362	11	22	32	43	54
50	$\bar{1}$.88425	88489	88552	88615	88678	88741	88803	88865	88927	88989	10	21	31	42	52
51	$\bar{1}$.89050	89112	89173	89233	89294	89354	89415	89475	89534	89594	10	20	30	40	50
52	$\bar{1}$.89653	89712	89771	89830	89888	89947	90005	90063	90120	90178	10	19	29	39	49
53	$\bar{1}$.90235	90292	90349	90405	90462	90518	90574	90630	90685	90741	9	19	28	37	47
54	$\bar{1}$.90796	90851	90905	90960	91014	91069	91123	91176	91230	91283	9	18	27	36	45
55	$\bar{1}$.91336	91389	91442	91495	91547	91599	91651	91703	91755	91806	9	17	26	35	43
56	$\bar{1}$.91857	91908	91959	92010	92060	92111	92161	92211	92260	92310	8	17	25	34	42
57	$\bar{1}$.92359	92408	92457	92506	92555	92603	92651	92699	92747	92795	8	16	24	32	40
58	$\bar{1}$.92842	92889	92936	92983	93030	93077	93123	93169	93215	93261	8	16	23	31	39
59	$\bar{1}$.93307	93352	93397	93442	93487	93532	93577	93621	93665	93709	7	15	22	30	37
60	$\bar{1}$.93753	93797	93840	93884	93927	93970	94012	94055	94098	94140	7	14	21	29	36
61	$\bar{1}$.94182	94224	94266	94307	94349	94390	94431	94472	94513	94553	7	14	21	27	34
62	$\bar{1}$.94593	94634	94674	94714	94753	94793	94832	94871	94911	94949	7	13	20	26	33
63	$\bar{1}$.94988	95027	95065	95103	95141	95179	95217	95254	95292	95329	6	13	19	25	32
64	$\bar{1}$.95366	95403	95440	95476	95513	95549	95585	95621	95657	95692	6	12	18	24	30
65	$\bar{1}$.95728	95763	95798	95833	95868	95902	95937	95971	96005	96039	6	12	17	23	29
66	$\bar{1}$.96073	96107	96140	96174	96207	96240	96273	96305	96338	96370	6	11	17	22	28
67	$\bar{1}$.96403	96435	96467	96498	96530	96562	96593	96624	96656	96686	5	10	16	21	26
68	$\bar{1}$.96717	96747	96778	96808	96838	96868	96898	96927	96957	96986	5	10	15	20	25
69	$\bar{1}$.97015	97044	97073	97102	97130	97159	97187	97215	97243	97271	5	9	14	19	24
70	$\bar{1}$.97299	97326	97353	97381	97408	97435	97461	97488	97515	97541	4	9	13	18	22
71	$\bar{1}$.97567	97593	97619	97645	97670	97696	97721	97746	97771	97796	4	8	13	17	21
72	$\bar{1}$.97821	97845	97870	97894	97918	97942	97966	97989	98013	98036	4	8	12	16	20
73	$\bar{1}$.98060	98083	98106	98128	98151	98174	98196	98218	98240	98262	4	7	11	15	19
74	$\bar{1}$.98284	98306	98327	98349	98370	98391	98412	98433	98453	98474	4	7	11	14	18
75	$\bar{1}$.98494	98515	98535	98555	98574	98594	98614	98633	98652	98671	3	7	10	13	16
76	$\bar{1}$.98690	98709	98728	98746	98765	98783	98801	98819	98837	98855	3	6	9	12	15
77	$\bar{1}$.98872	98890	98907	98924	98941	98958	98975	98991	99008	99024	3	6	8	11	14
78	$\bar{1}$.99040	99056	99072	99088	99104	99119	99135	99150	99165	99180	3	5	8	10	13
79	$\bar{1}$.99195	99209	99224	99238	99253	99267	99281	99294	99308	99322	2	5	7	9	12
80	$\bar{1}$.99335	99348	99362	99375	99388	99400	99413	99425	99438	99450	2	4	6	8	11
81	$\bar{1}$.99462	99474	99486	99497	99509	99520	99532	99543	99554	99565	2	4	6	8	9
82	$\bar{1}$.99575	99586	99596	99607	99617	99627	99637	99647	99657	99666	2	3	5	7	8
83	$\bar{1}$.99675	99684	99693	99702	99711	99720	99728	99737	99745	99753	1	3	4	6	7
84	$\bar{1}$.99761	99769	99777	99785	99792	99800	99807	99814	99821	99828	1	2	4	5	8
85	$\bar{1}$.99834	99841	99847	99854	99860	99866	99872	99878	99883	99889	1	2	3	4	5
86	$\bar{1}$.99894	99899	99904	99909	99914	99919	99923	99928	99932	99936	1	2	2	3	4
87	$\bar{1}$.99940	99944	99948	99952	99955	99959	99962	99965	99968	99971	1	1	2	2	3
88	$\bar{1}$.99974	99976	99979	99981	99983	99985	99987	99989	99990	99992	0	1	1	1	2
89	$\bar{1}$.99993	99995	99996	99997	99998	99999	99999	99999	0000	0000	0	0	0	0	1
90	0.0000														

LOGARITHMS OF COSINES.

Numbers in difference columns to be subtracted, not added

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	0.00000	00000	00000	1.99999	99999	99998	99998	99997	99996	99995	0	0	0	0	1
1	$\bar{1}$.99993	99992	99990	99989	99987	99985	99983	99981	99979	99976	0	1	1	1	2
2	$\bar{1}$.99974	99971	99968	99965	99962	99959	99955	99952	99948	99944	1	1	2	2	3
3	$\bar{1}$.99940	99936	99932	99928	99923	99919	99914	99909	99904	99899	1	2	2	3	4
4	$\bar{1}$.99894	99889	99883	99878	99872	99866	99860	99854	99847	99841	1	2	3	4	5
5	$\bar{1}$.99834	99828	99821	99814	99807	99800	99792	99785	99777	99769	1	2	4	5	6
6	$\bar{1}$.99761	99753	99745	99737	99728	99720	99711	99702	99693	99684	1	3	4	6	7
7	$\bar{1}$.99675	99666	99656	99647	99637	99627	99617	99607	99596	99586	2	3	5	7	8
8	$\bar{1}$.99575	99565	99554	99543	99532	99520	99509	99497	99486	99474	2	4	6	8	9
9	$\bar{1}$.99462	99450	99438	99425	99413	99400	99388	99375	99362	99348	2	4	6	8	11
10	$\bar{1}$.99335	99322	99308	99294	99281	99267	99253	99238	99224	99209	2	5	7	9	12
11	$\bar{1}$.99195	99180	99165	99150	99135	99119	99104	99088	99072	99056	3	5	8	10	13
12	$\bar{1}$.99040	99024	99008	98991	98975	98958	98941	98924	98907	98890	3	6	8	11	14
13	$\bar{1}$.98872	98855	98837	98819	98801	98783	98765	98746	98728	98709	3	6	9	12	15
14	$\bar{1}$.98690	98671	98652	98633	98614	98594	98574	98555	98535	98515	3	7	10	13	16
15	$\bar{1}$.98494	98474	98453	98433	98412	98391	98370	98349	98327	98306	4	7	11	14	18
16	$\bar{1}$.98284	98262	98240	98218	98196	98174	98151	98128	98106	98083	4	7	11	15	19
17	$\bar{1}$.98061	98036	98013	97989	97966	97942	97918	97894	97870	97845	4	8	12	16	20
18	$\bar{1}$.97821	97796	97771	97746	97721	97696	97670	97645	97619	97593	4	8	13	17	21
19	$\bar{1}$.97567	97541	97515	97488	97461	97435	97408	97381	97353	97326	4	9	13	18	22
20	$\bar{1}$.97299	97271	97243	97215	97187	97159	97130	97102	97073	97044	5	9	14	19	24
21	$\bar{1}$.97015	96986	96957	96927	96898	96868	96838	96808	96778	96747	5	10	15	20	25
22	$\bar{1}$.96717	96686	96655	96624	96593	96562	96530	96498	96467	96435	5	10	16	21	26
23	$\bar{1}$.96403	96370	96338	96305	96273	96240	96207	96174	96140	96107	6	11	17	22	28
24	$\bar{1}$.96073	96039	96005	95971	95937	95902	95868	95833	95798	95763	6	12	17	23	29
25	$\bar{1}$.95728	95692	95657	95621	95585	95549	95513	95476	95440	95403	6	12	18	24	30
26	$\bar{1}$.95366	95329	95292	95254	95217	95179	95141	95103	95065	95027	6	13	19	25	32
27	$\bar{1}$.94988	94949	94911	94871	94832	94793	94753	94714	94674	94634	7	13	20	26	33
28	$\bar{1}$.94593	94553	94513	94472	94431	94390	94349	94307	94266	94224	7	14	21	27	34
29	$\bar{1}$.94182	94140	94098	94055	94012	93970	93927	93884	93840	93797	7	14	21	29	36
30	$\bar{1}$.93753	93709	93665	93621	93577	93532	93487	93442	93397	93352	7	15	22	30	37
31	$\bar{1}$.93307	93261	93215	93169	93123	93077	93030	92983	92936	92889	8	16	23	31	39
32	$\bar{1}$.92842	92795	92747	92699	92651	92603	92555	92506	92457	92408	8	16	24	32	40
33	$\bar{1}$.92360	92310	92260	92211	92161	92111	92060	92010	91959	91908	8	17	25	34	42
34	$\bar{1}$.91857	91806	91755	91703	91651	91599	91547	91495	91442	91389	9	17	26	35	43
35	$\bar{1}$.91336	91283	91230	91176	91123	91069	91014	90960	90905	90851	9	18	27	36	45
36	$\bar{1}$.90796	90741	90685	90630	90574	90518	90462	90405	90349	90292	9	19	28	37	47
37	$\bar{1}$.90235	90178	90120	90063	90005	89947	89888	89830	89771	89712	10	19	29	39	49
38	$\bar{1}$.89653	89594	89534	89475	89415	89354	89294	89233	89173	89112	10	20	30	40	50
39	$\bar{1}$.89050	88989	88927	88865	88803	88741	88678	88615	88552	88489	10	21	31	42	52
40	$\bar{1}$.88425	88362	88298	88234	88169	88105	88040	87975	87909	87844	11	22	32	43	54
41	$\bar{1}$.87778	87712	87646	87579	87513	87446	87378	87311	87243	87175	11	22	34	45	56
42	$\bar{1}$.87107	87039	86970	86902	86832	86763	86694	86624	86554	86483	12	23	35	46	58
43	$\bar{1}$.86413	86342	86271	86200	86128	86056	85984	85912	85839	85766	12	24	36	48	60
44	$\bar{1}$.85693	85620	85546	85473	85399	85324	85250	85075	85100	85024	12	25	37	50	62

LOGARITHMS OF COSINES.

Numbers in difference columns to be subtracted, not added

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	$\bar{1}$.84948	84873	84796	84720	84643	84566	84489	84411	84334	84255	12	26	39	51	64
46	$\bar{1}$.84177	84098	84020	83940	83861	83781	83701	83621	83540	83459	13	27	40	53	67
47	$\bar{1}$.83378	83297	83215	83133	83051	82968	82885	82802	82719	82635	14	28	41	55	69
48	$\bar{1}$.82551	82467	82382	82297	82212	82126	82041	81955	81868	81781	14	29	43	57	71
49	$\bar{1}$.81694	81607	81519	81431	81343	81254	81166	81076	80987	80897	15	30	44	59	74
50	$\bar{1}$.80807	80716	80625	80534	80443	80351	80259	80166	80074	79981	15	31	46	61	77
51	$\bar{1}$.79887	79793	79699	79605	79510	79415	79319	79224	79128	79031	16	32	48	64	79
52	$\bar{1}$.78934	78837	78739	78642	78543	78445	78346	78246	78147	78047	16	33	49	66	82
53	$\bar{1}$.77946	77846	77744	77643	77541	77439	77336	77233	77130	77026	17	34	51	68	85
54	$\bar{1}$.76922	76817	76712	76607	76501	76395	76289	76182	76075	75967	18	35	53	71	89
55	$\bar{1}$.75869	75751	75642	75533	75423	75313	75202	75091	74980	74868	18	37	55	74	92
56	$\bar{1}$.74756	74644	74531	74417	74303	74189	74074	73959	73843	73727	19	38	57	76	95
57	$\bar{1}$.73611	73494	73377	73259	73140	73022	72902	72783	72663	72542	20	40	60	79	99
58	$\bar{1}$.72421	72299	72177	72055	71932	71809	71685	71560	71435	71310	21	41	62	82	103
59	$\bar{1}$.71184	71058	70931	70803	70675	70547	70418	70288	70159	70028	21	43	64	86	107
60	$\bar{1}$.69897	69765	69633	69501	69368	69234	69100	68965	68829	68694	23	45	67	89	112
61	$\bar{1}$.68557	68420	68282	68144	68006	67866	67726	67586	67445	67303	24	47	70	93	116
62	$\bar{1}$.67161	67018	66875	66731	66586	66441	66295	66148	66001	65853	25	49	73	97	121
63	$\bar{1}$.65705	65556	65406	65255	65104	64953	64800	64647	64494	64339	26	51	77	101	126
64	$\bar{1}$.64184	64028	63872	63715	63557	63398	63239	63079	62918	62757	27	53	80	106	132
65	$\bar{1}$.62595	62432	62268	62104	61939	61773	61606	61438	61270	61101	28	56	83	111	138
66	$\bar{1}$.60931	60761	60589	60417	60244	60070	59895	59720	59543	59366	29	58	87	116	145
67	$\bar{1}$.59188	59009	58829	58648	58467	58284	58101	57916	57731	57545	31	61	91	122	152
68	$\bar{1}$.57358	57169	56980	56790	56599	56408	56215	56021	55826	55630	33	64	96	128	160
69	$\bar{1}$.55433	55235	55036	54836	54635	54433	54229	54025	53819	53613	34	68	101	135	169
70	$\bar{1}$.53405	53196	52986	52775	52563	52350	52135	51919	51702	51484	35	71	107	143	178
71	$\bar{1}$.51264	51043	50821	50598	50374	50148	49920	49692	49462	49231	37	75	113	151	189
72	$\bar{1}$.48998	48764	48529	48292	48054	47814	47573	47330	47086	46841	40	80	120	160	200
73	$\bar{1}$.46594	46345	46095	45843	75589	45334	45077	44819	44559	44297	42	85	128	170	213
74	$\bar{1}$.44034	43769	43502	43233	42962	42690	42416	42139	41861	41582	45	91	136	182	228
75	$\bar{1}$.41300	41016	40730	40442	40152	39860	39566	39270	38971	38670	49	98	146	195	244
76	$\bar{1}$.38368	38062	37755	37445	37133	36819	36502	36182	35860	35536	53	105	158	211	263
77	$\bar{1}$.35209	34879	34547	34212	33874	33534	33190	32844	32495	32143	57	114	171	228	285
78	$\bar{1}$.31788	31430	31068	30704	30336	29966	29591	29214	28833	28448	62	124	186	248	311
79	$\bar{1}$.28060	27668	27273	26873	26470	26063	25652	25237	24818	24395	68	136	205	273	341
80	$\bar{1}$.23967	23535	23098	22657	22211	21761	21306	20845	20380	19909	76	151	227	302	378
81	$\bar{1}$.19433	18952	18465	17973	17474	16970	16460	15944	15421	14891	84	169	253	338	423
82	$\bar{1}$.14356	13813	13263	12706	12142	11570	10990	10402	09807	09202	96	192	288	384	480
83	$\bar{1}$.08589	07968	07337	06696	06046	05386	04715	04034	03342	02639	111	222	333	444	555
84	$\bar{1}$.01923	01196	00456	$\bar{2}$.99704	$\bar{2}$.98937	$\bar{2}$.98157	$\bar{2}$.97363	$\bar{2}$.96553	$\bar{2}$.95728	$\bar{2}$.94887	132	263	395	526	658
85	$\bar{2}$.94030	93154	92261	91349	90417	89464	88490	87494	86474	85429					
86	$\bar{2}$.84358	83261	82134	80978	79789	78566	77310	76015	74680	73303					
87	$\bar{2}$.71880	70409	68886	67308	65670	63968	62196	60349	58419	56400					
88	$\bar{2}$.54282	52055	49708	47226	44594	41792	38796	35578	32103	28324					
89	$\bar{2}$.24186	19610	14495	08696	02002	$\bar{3}$.94084	$\bar{3}$.84393	$\bar{3}$.71900	$\bar{3}$.54291	$\bar{3}$.24188					
90	00														

LOGARITHMS OF TANGENTS.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	−∞	3.24188	3.54291	3.71900	3.84394	3.94086	2.02004	2.08700	2.14500	2.19616					
1	2.24192	28332	32112	35590	38809	41807	44611	47245	49729	52079					
2	2.54308	56429	58451	60384	62234	64009	65715	67356	68938	70465					
3	2.71940	73366	74748	76087	77387	78649	79875	81068	82230	83361					
4	2.84464	85540	86591	87616	88618	89598	90557	91495	92414	93313					
5	2.94195	95060	95908	96739	97556	98358	99145	99919	1.00679	1.01427	133	266	398	531	664
6	1.02162	02885	03597	04297	04987	05666	06335	06994	07643	08283	112	225	337	450	562
7	1.08914	09537	10150	10756	11353	11943	12525	13099	13667	14227	98	195	293	391	489
8	1.14780	15327	15867	16401	16928	17450	17965	18475	18979	19478	86	173	259	346	432
9	1.19971	20459	20942	21420	21893	22361	22824	23283	23737	24186	78	155	233	311	388
10	1.24632	25073	25510	25943	26372	26797	27218	27635	28049	28459	70	141	212	282	353
11	1.28865	29268	29668	30064	30457	30846	31233	31616	31996	32373	64	129	194	258	323
12	1.32747	33119	33487	33853	34215	34576	34933	35288	35640	35989	60	119	179	239	299
13	1.36336	36681	37023	37363	37700	38035	38368	38699	39027	39353	56	111	167	223	278
14	1.39677	39999	40319	40636	40952	41266	41578	41887	42195	42501	52	104	156	208	261
15	1.42805	43108	43408	43707	44004	44299	44592	44884	45174	45463	49	98	147	196	244
16	1.45750	46035	46319	46601	46881	47160	47438	47714	47989	48262	46	93	139	185	232
17	1.48534	48804	49073	49341	49607	49872	50136	50398	50659	50919	44	88	132	176	220
18	1.51178	51435	51691	51946	52200	52452	52703	52953	53202	53450	42	84	126	168	210
19	1.53697	53943	54187	54431	54673	54915	55155	55395	55633	55870	40	80	120	160	201
20	1.56107	56342	56576	56810	57042	57274	57504	57734	57963	58191	39	77	116	154	193
21	1.58418	58644	58869	59094	59317	59540	59762	59983	60203	60422	37	74	111	148	185
22	1.60641	60859	61076	61292	61508	61722	61936	62150	62362	62574	36	71	107	143	179
23	1.62785	62996	63205	63414	63623	63830	64037	64243	64449	64654	35	69	104	138	173
24	1.64858	65062	65265	65467	65669	65870	66071	66271	66470	66669	33	67	100	134	169
25	1.66867	67065	67262	67458	67654	67850	68044	68239	68432	68626	33	65	98	130	163
26	1.68818	69010	69202	69393	69584	59774	69963	70152	70341	70529	32	63	95	127	158
27	1.70717	70904	71090	71277	71462	71648	71833	72017	72201	72384	31	62	93	123	154
28	1.72567	72750	72932	73114	73295	73476	73657	73837	74017	74196	30	60	90	121	151
29	1.74375	74554	74732	74910	75087	75264	75441	75617	75793	75969	29	59	88	118	147
30	1.76144	76319	76493	76668	76841	77015	77188	77361	77533	77706	29	58	87	116	144
31	1.77877	78049	78220	78391	78562	78732	78902	79072	79241	79410	28	57	85	113	142
32	1.79579	79747	79916	80084	80251	80419	80586	80753	80919	81086	28	56	84	112	139
33	1.81252	81418	81583	81748	81913	82078	82243	82407	82571	82735	27	55	82	110	137
34	1.82899	83062	83225	83388	83551	83713	83876	84038	84200	84361	27	54	81	108	135
35	1.84523	84684	84845	85006	85166	85327	85487	85647	85807	85967	27	53	80	107	134
36	1.86126	86285	86445	86603	86762	86921	87079	87238	87396	87554	26	53	79	106	132
37	1.87711	87869	88027	88184	88341	88498	88655	88812	88968	89125	26	52	79	105	131
38	1.89281	89437	89593	89749	89905	90061	90216	90371	90527	90682	26	52	78	104	130
39	1.90837	90992	91147	91301	91456	91610	91765	91919	92073	92227	26	51	77	103	129
40	1.92381	92535	92689	92843	92996	93150	93303	93457	93610	93763	26	51	77	102	128
41	1.93916	94069	94222	94375	94528	94681	94834	94986	95139	95291	26	51	76	102	127
42	1.95444	95596	95748	95901	96053	96205	96357	96510	96662	96814	25	51	76	101	127
43	1.96966	97118	97269	97421	97573	97725	97877	98029	98180	98332	25	51	76	101	126
44	1.98484	98635	98787	98939	99090	99242	99394	99545	99697	99848	25	51	76	101	126

LOGARITHMS OF TANGENTS.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	.0000	00152	00303	00455	00606	00758	00910	01061	01213	01365	25	51	76	101	126
46	.01516	01668	01820	01971	02123	02275	02427	02579	02731	02882	25	51	76	101	126
47	.03034	03186	03338	03490	03643	03795	03947	04099	04251	04404	25	51	76	101	127
48	.04556	04709	04861	05014	05166	05319	05472	05625	05778	05931	25	51	76	102	127
49	.06084	06237	06390	06543	06697	06850	07004	07157	07311	07465	26	51	77	102	128
50	.07619	07773	07927	08081	08235	08390	08544	08699	08853	09008	26	51	77	103	129
51	.09163	09318	09473	09629	09784	09939	10095	10251	10407	10563	26	52	78	104	130
52	.10719	10875	11032	11188	11345	11502	11659	11816	11973	12131	26	52	79	105	131
53	.12289	12446	12604	12762	12921	13079	13238	13396	13555	13715	26	53	79	106	132
54	.13874	14033	14193	14353	14513	14673	14834	14994	15155	15316	27	53	80	107	134
55	.15477	15639	15800	15962	16124	16287	16449	16612	16775	16938	27	54	81	108	135
56	.17101	17265	17429	17593	17757	17922	18087	18252	18417	18582	27	55	82	110	137
57	.18748	18914	19081	19247	19414	19581	19749	19916	20084	20253	28	56	84	112	139
58	.20421	20590	20759	20928	21098	21268	21438	21609	21780	21951	28	57	85	113	142
59	.22123	22294	22467	22639	22812	22985	23159	23332	23507	23681	29	58	87	116	144
60	.23856	24031	24207	24383	24559	24736	24913	25090	25268	25446	29	59	88	118	147
61	.25625	25804	25983	26163	26343	26524	26705	26886	27068	27250	30	60	90	121	151
62	.27433	27616	27799	27983	28167	28352	28538	28723	28910	29096	31	62	93	123	154
63	.29283	29471	29659	29848	30037	30226	30416	30607	30798	30990	32	63	95	127	158
64	.31182	31374	31568	31761	31956	32150	32346	32542	32738	32935	33	65	98	130	163
65	.33133	33331	33530	33729	33929	34130	34331	34533	34735	34938	33	66	100	134	167
66	.35142	35346	35551	35757	35963	36170	36377	36586	36795	37004	34	69	104	138	172
67	.37215	37426	37638	37850	38064	38278	38492	38708	38924	39141	35	71	107	143	178
68	.39359	39578	39797	40017	40238	40460	40683	40906	41131	41356	37	74	111	148	185
69	.41582	41809	42037	42266	42496	42726	42958	43190	43424	43658	38	77	115	154	193
70	.43893	44130	44367	44605	44845	45085	45327	45569	45813	46057	40	80	120	160	201
71	.46303	46550	46797	47047	47297	47548	47800	48054	48309	48565	42	84	126	168	210
72	.48822	49081	49341	49602	49864	50128	50393	50659	50927	51196	44	88	132	176	220
73	.51466	51738	52011	52286	52562	52840	53119	53399	53681	53965	46	93	139	185	232
74	.54250	54537	54826	55116	55408	55701	55996	56293	56592	56892	49	98	147	195	245
75	.57195	57499	57805	58113	58422	58734	59048	59364	59681	60001	52	104	156	208	261
76	.60323	60647	60973	61301	61632	61965	62300	62637	62977	63319	56	111	167	223	278
77	.63664	64011	64360	64712	65067	65424	65785	66147	66513	66881	60	119	178	239	298
78	.67253	67627	68004	68384	68767	69154	69543	69936	70332	70732	64	129	194	258	323
79	.71135	71541	71951	72365	72782	73203	73628	74057	74490	74927	70	141	212	282	353
80	.75368	75813	76263	76717	77176	77639	78107	78580	79058	79541	78	155	233	316	388
81	.80029	80522	81021	81525	82035	82550	83072	83599	84133	84673	86	173	259	346	432
82	.85220	85773	86333	86901	87475	88057	88647	89244	89850	90463	98	195	293	391	489
83	.91086	91717	92357	93006	93665	94334	95013	95703	96403	97115	112	225	337	450	563
84	.97838	98573	99321	1.00081	1.00855	1.01642	1.02444	1.03261	1.04092	1.04940	133	266	398	531	664
85	1.05805	06687	07586	08505	09443	10402	11381	12384	13409	14460					
86	1.15536	16639	17770	18932	20125	21351	22613	23913	25252	26634					
87	1.28060	29535	31062	32644	34285	35991	37766	39616	41549	43571					
88	1.45692	47921	50271	52755	55389	58193	61191	64410	67888	71668					
89	1.75808	80384	85500	91300	97996	2.05914	2.15606	2.28100	2.45710	2.75812					

LOGARITHMS OF COSECANTS.

Numbers in difference columns to be subtracted, not added.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	∞	2.7581	45709	28100	15607	05916	1.97998	1.91304	1.85505	1.80390					
1	1.75814	71676	67897	64422	61204	58208	55406	52774	50292	47945					
2	1.45718	43600	41581	39651	37804	36032	34330	32692	31114	29591					
3	1.28120	26697	25320	23985	22690	21432	20211	19022	17866	16739					
4	1.15642	14571	13526	12506	11510	10536	09583	08651	07739	06846					
5	1.05970	05113	04272	03447	02637	01843	01063	00296	0.99544	0.98804					
6	.98077	97361	96658	95966	95285	94614	93954	93304	92663	92032	111	222	333	444	556
7	.91411	90798	90193	89598	89010	88430	87858	87294	86737	86187	96	192	288	384	480
8	.85644	85109	84579	84056	83540	83030	82526	82027	81535	81048	84	169	253	338	423
9	.80567	80091	79620	79155	78694	78239	77789	77343	76902	76465	76	151	227	302	378
10	.76033	75605	75182	74763	74348	73937	73530	73127	72727	72332	68	136	205	272	341
11	.71940	71552	71167	70786	70409	70034	69664	69296	68932	68570	62	124	186	248	311
12	.68212	67857	67505	67156	66810	66466	66126	65788	65453	65121	57	114	171	228	285
13	.64791	64464	64140	63818	63498	63181	62867	62555	62245	61938	53	105	158	211	263
14	.61632	61330	61029	60730	60434	60140	59848	59558	59270	58984	49	98	146	195	244
15	.58700	58418	58139	57860	57584	57310	57038	56767	56498	56231	46	91	136	182	228
16	.55966	55703	55441	55181	54923	54666	54411	54157	53905	53655	42	85	128	170	213
17	.53406	53159	52914	52670	52427	52186	51946	51708	51471	51236	40	80	120	160	200
18	.51002	50769	50538	50308	50080	49852	49626	49402	49179	48957	37	75	113	151	189
19	.48736	48516	48298	48081	47865	47650	47437	47225	47014	46804	35	71	107	143	178
20	.46595	46387	46181	45975	45771	45567	45365	45164	44964	44765	33	68	101	135	169
21	.44567	44370	44174	43979	43785	43592	43401	43210	43020	42831	32	64	96	128	160
22	.42642	42455	42269	42084	41899	41716	41533	41352	41171	40991	30	61	91	122	152
23	.40812	40634	40457	40280	40105	39930	39756	39583	39411	39239	29	58	87	116	145
24	.39069	38899	38730	38561	38394	38227	38061	37896	37732	37568	27	56	83	111	138
25	.37405	37243	37082	36921	36761	36602	36443	36285	36128	35972	26	53	79	106	132
26	.35816	35661	35506	35353	35200	35047	34896	34745	34594	34444	25	51	76	101	127
27	.34295	34147	33999	33852	33705	33559	33414	33269	33125	32982	24	49	73	97	121
28	.32839	32697	32556	32414	32274	32134	31994	31856	31717	31580	23	47	70	93	116
29	.31443	31306	31171	31035	30900	30766	30632	30499	30367	30235	22	45	67	89	112
30	.30103	29972	29841	29712	29582	29453	29325	29197	29069	28942	21	43	64	86	107
31	.28816	28690	28565	28440	28315	28191	28068	27945	27823	27701	21	41	62	82	103
32	.27579	27458	27337	27217	27098	26978	26860	26741	26623	26506	20	40	60	79	99
33	.26389	26273	26157	26041	25926	25811	25697	25583	25469	25356	19	38	57	76	95
34	.25244	25132	25020	24909	24798	24687	24577	24467	24358	24249	18	37	55	74	92
35	.24141	24033	23925	23818	23711	23605	23499	23393	23288	23183	18	35	53	71	89
36	.23078	22974	22870	22767	22664	22561	22459	22357	22256	22154	17	34	51	68	85
37	.22054	21953	21853	21754	21654	21555	21457	21358	21261	21163	16	33	49	66	82
38	.21066	20969	20872	20776	20681	20585	20490	20395	20301	20207	16	32	48	64	79
39	.20113	20019	19926	19834	19741	19649	19557	19466	19375	19284	15	31	46	61	77
40	.19193	19103	19013	18924	18834	18746	18657	18569	18481	18393	15	30	44	59	74
41	.18306	18219	18132	18046	17959	17874	17788	17703	17618	17533	14	29	43	57	71
42	.17449	17365	17281	17198	17115	17032	16949	16867	16785	16703	14	28	41	55	69
43	.16622	16541	16460	16379	16299	16219	16139	16060	15980	15901	13	27	40	53	67
44	.15823	15745	15666	15589	15511	15434	15357	15280	15204	15127	13	26	39	51	64

LOGARITHMS OF COSECANTS

Numbers in difference columns to be subtracted, not added.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	.15051	14976	14900	14825	14750	14676	14601	14527	14453	14380	12	25	37	50	62
46	.14307	14234	14161	14088	14016	13944	13872	13800	13729	13658	12	24	36	48	60
47	.13587	13517	13446	13376	13306	13237	13168	13098	13030	12961	12	23	35	46	58
48	.12893	12825	12757	12689	12622	12554	12487	12421	12354	12288	11	22	34	45	56
49	.12222	12156	12091	12025	11960	11895	11831	11766	11702	11638	11	22	32	43	54
50	.11575	11511	11448	11385	11322	11259	11197	11135	11073	11011	10	21	31	42	52
51	.10950	10888	10827	10767	10706	10646	10585	10525	10466	10406	10	20	30	40	50
52	.10347	10288	10229	10170	10112	10053	09995	09937	09880	09822	10	19	29	39	49
53	.09765	09708	09651	09595	09538	09482	09426	09370	09315	09259	9	19	28	37	47
54	.09204	09149	09094	09040	08986	08931	08877	08824	08770	08717	9	18	27	36	45
55	.08664	08611	08558	08505	08453	08401	08349	08297	08245	08194	9	17	26	35	43
56	.08143	08092	08041	07990	07940	07889	07839	07789	07740	07690	8	17	25	33	42
57	.07641	07592	07543	07494	07445	07397	07349	07301	07253	07205	8	16	24	32	40
58	.07158	07111	07064	07017	06970	06923	06877	06831	06785	06739	8	16	23	31	39
59	.06693	06648	06603	06558	06513	06468	06423	06379	06335	06291	7	15	22	30	37
60	.06247	06203	06160	06116	06073	06030	05988	05945	05902	05860	7	14	21	29	36
61	.05818	05776	05734	05693	05651	05610	05569	05528	05487	05447	7	14	21	27	34
62	.05407	05366	05326	05286	05247	05207	05168	05129	05089	05051	7	13	20	26	33
63	.05012	04973	04935	04897	04859	04821	04783	04746	04708	04671	6	13	19	25	33
64	.04634	04597	04560	04524	04487	04451	04415	04379	04343	04308	6	12	18	24	30
65	.04272	04237	04202	04167	04132	04098	04063	04029	03995	03961	6	12	17	23	29
66	.03927	03893	03860	03826	03793	03760	03727	03695	03662	03630	6	11	17	22	28
67	.03597	03565	03533	03502	03470	03438	03407	03376	03345	03314	5	10	16	21	26
68	.03283	03253	03222	03192	03162	03132	03102	03073	03043	03014	5	10	15	20	25
69	.02985	02956	02927	02898	02870	02841	02813	02785	02757	02729	5	9	14	19	24
70	.02701	02674	02647	02619	02592	02565	02539	02512	02485	02459	4	9	13	18	22
71	.02433	02407	02381	02355	02330	02304	02279	02254	02229	02204	4	9	13	17	21
72	.02179	02155	02130	02106	02082	02058	02034	02011	01987	01964	4	8	12	16	20
73	.01940	01917	01894	01871	01849	01826	01804	01782	01760	01738	4	7	11	15	19
74	.01716	01694	01673	01651	01630	01609	01588	01567	01547	01526	4	7	11	14	18
75	.01506	01485	01465	01445	01426	01406	01386	01367	01348	01329	3	7	10	13	16
76	.01310	01291	01272	01254	01235	01217	01199	01181	01163	01145	3	6	9	12	15
77	.01128	01110	01093	01076	01059	01042	01025	01009	00992	00976	3	6	8	11	14
78	.00960	00944	00928	00912	00896	00881	00865	00850	00835	00820	3	5	8	10	13
79	.00805	00791	00776	00762	00747	00733	00719	00706	00692	00678	2	5	7	9	11
80	.00665	00652	00638	00625	00612	00600	00587	00575	00562	00550	2	4	6	8	11
81	.00538	00526	00514	00503	00491	00480	00468	00457	00446	00435	2	4	6	8	9
82	.00425	00414	00404	00393	00383	00373	00363	00353	00344	00334	2	3	5	7	8
83	.00325	00316	00307	00298	00289	00280	00271	00263	00255	00247	1	3	4	6	7
84	.00239	00231	00223	00215	00208	00200	00193	00186	00179	00172	1	2	4	5	6
85	.00166	00159	00153	00146	00140	00134	00128	00122	00117	00111	1	2	3	4	5
86	.00106	00101	00096	00091	00086	00081	00077	00072	00068	00064	1	2	2	3	4
87	.00060	00056	00052	00048	00045	00041	00038	00035	00032	00029	1	1	2	2	3
88	.00026	00024	00021	00019	00017	00015	00013	00011	00010	00008	0	1	1	1	2
89	.00007	00005	00004	00003	00002	00002	00001	00001	00000	00000	0	0	0	0	1

LOGARITHMS OF SECANTS.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	.00000	00000	00000	00001	00001	00002	00002	00003	00004	00005	0	0	0	0	1
1	.00007	00008	00010	00011	00013	00015	00017	00019	00021	00024	0	1	1	1	2
2	.00026	00029	00032	00035	00038	00041	00045	00048	00052	00056	1	1	2	2	3
3	.00060	00064	00068	00072	00077	00081	00086	00091	00096	00101	1	2	2	3	4
4	.000106	00111	00117	00122	00128	00134	00140	00146	00153	00159	1	2	3	4	5
5	.00166	00172	00179	00186	00193	00200	00208	00215	00223	00231	1	2	4	5	6
6	.00239	00247	00255	00263	00271	00280	00289	00298	00307	00316	1	3	4	6	7
7	.00325	00334	00344	00353	00363	00373	00383	00393	00404	00414	2	3	5	7	8
8	.00425	00435	00446	00457	00468	00480	00491	00503	00514	00526	2	4	6	8	9
9	.00538	00550	00562	00575	00587	00600	00612	00625	00638	00652	2	5	7	9	11
10	.00665	00678	00692	00706	00719	00733	00747	00762	00776	00791	2	5	7	9	11
11	.00805	00820	00835	00850	00865	00881	00896	00912	00928	00944	3	5	8	10	13
12	.00960	00976	00992	01009	01025	01042	01059	01076	01093	01110	3	6	8	11	14
13	.01128	01145	01163	01181	01199	01217	01235	01254	01272	01291	3	6	9	12	15
14	.01310	01329	01348	01367	01386	01406	01426	01445	01465	01485	3	7	10	13	16
15	.01506	01526	01547	01567	01588	01609	01630	01651	01673	01694	4	7	11	14	18
16	.01716	01738	01760	01782	01804	01826	01849	01871	01894	01947	4	7	11	15	19
17	.01940	01964	01987	02011	02034	02058	02082	02106	02130	02155	4	8	12	16	20
18	.02179	02204	02229	02254	02279	02304	02330	02355	02381	02407	4	9	13	17	21
19	.02433	02459	02485	02512	02539	02565	02592	02619	02647	02674	4	9	13	18	22
20	.02701	02729	02757	02785	02813	02841	02870	02898	02927	02956	5	9	14	19	24
21	.02985	03014	03043	03073	03102	03132	03162	03192	03222	03253	5	10	15	20	25
22	.03283	03314	03345	03376	03407	03438	03470	03502	03533	03565	5	10	16	21	26
23	.03597	03630	03662	03695	03727	03760	03793	03826	03860	03893	6	11	17	22	28
24	.03927	03961	03995	04029	04063	04098	04132	04167	04202	04237	6	12	17	23	29
25	.04272	04308	04343	04379	04415	04451	04487	05424	04560	04597	6	12	18	24	30
26	.04634	04671	04708	04746	04783	04821	04859	04897	04935	04973	6	13	19	25	32
27	.05012	05051	05089	05129	05168	05207	05247	05286	05326	05366	7	13	20	26	33
28	.05402	05447	05487	05528	05569	05610	05651	05693	05734	05776	7	14	21	27	34
29	.05818	05860	05902	05945	05988	06030	06073	06116	06160	06203	7	14	21	29	36
30	.06247	06291	06335	06379	06423	06468	06513	06558	06603	06648	7	15	22	30	27
31	.06693	06739	06785	06831	06877	06923	06970	07017	07064	07111	8	16	23	31	39
32	.07158	07205	07253	07301	07349	07397	07445	07494	07543	07592	8	16	24	32	40
33	.07641	07690	07740	07789	07839	07889	07940	07990	08041	08092	8	17	25	33	42
34	.08143	08194	08245	08297	08349	08401	08453	08505	08558	08611	9	17	26	35	43
35	.08664	08717	08770	08824	08877	08931	08986	09040	09094	09149	9	18	27	36	45
36	.09204	09259	09315	09370	09426	09482	09538	09595	09651	09708	9	19	28	37	47
37	.09765	09822	09880	09937	09995	10053	10112	10170	10229	10288	10	19	29	39	49
38	.10347	10406	10466	10525	10585	10646	10706	10767	10827	10888	10	20	30	40	50
39	.10950	11011	11073	11135	11197	11259	11322	11385	11448	11511	10	21	31	42	52
40	.11575	11638	11702	11766	11831	11895	11960	12025	12091	12156	11	22	32	43	54
41	.12222	12288	12354	12421	12487	12554	12622	12689	12757	12825	11	22	34	45	56
42	.12893	12961	13030	13098	13168	13237	13306	13376	13446	13517	12	23	35	46	58
43	.13587	13658	13729	13800	13872	13944	14016	14088	14161	14234	12	24	36	48	60
44	.14307	14380	14454	14527	14601	14676	14750	14525	14900	14976	12	25	37	50	62

LOGARITHMS OF SECANTS.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	.15051	15127	15204	15280	15357	15434	15511	15589	15666	15745	13	26	39	51	64
46	15823	15901	15980	16060	16139	16219	16299	16379	16460	16541	13	27	40	53	67
47	16622	16703	16785	16867	16949	17032	17115	17198	17281	17365	14	28	41	55	69
48	17449	17533	17618	17703	17788	17874	17959	18046	18132	18219	14	29	43	57	71
49	18306	18393	18481	18569	18657	18746	18834	18924	19013	19103	15	30	44	59	74
50	19193	19284	19375	19466	19557	19649	19741	19834	19926	20019	15	31	46	61	77
51	20113	20207	20301	20395	20490	20585	20681	20776	20872	20969	16	32	48	64	79
52	21066	21163	21261	21358	21457	21555	21654	21754	21853	21953	16	33	49	66	82
53	22054	22154	22256	22357	22459	22561	22664	22767	22870	22974	17	34	51	68	85
54	23078	23183	23288	23393	23499	23605	23711	23818	23925	24033	18	35	53	71	89
55	24141	24249	24358	24467	24577	24687	24798	24909	25020	25132	18	37	55	74	92
56	25244	25356	25469	25583	25697	25811	25926	26041	26157	26273	19	38	57	76	95
57	26389	26506	26623	26741	26860	26978	27098	27217	27337	27458	20	40	60	79	99
58	27579	27701	27823	27945	28068	28191	28315	28440	28565	28690	21	41	62	82	103
59	28816	28942	29069	29197	29325	29453	29582	29712	29841	29972	21	43	64	86	107
60	30103	30235	30367	30499	30632	30766	30900	31035	31171	31306	22	45	67	89	112
61	31443	31580	31717	31856	31994	32134	32274	32414	32555	32697	23	47	70	93	116
62	32839	32982	33125	33269	33414	33559	33705	33852	33999	34147	24	49	73	97	121
63	34295	34444	34594	34745	34896	35047	35200	35353	35506	35661	25	51	76	101	127
64	35816	35972	36128	36285	36443	36602	36761	36921	37082	37243	26	53	79	106	132
65	37405	37568	37732	37896	38061	38227	38394	38561	38730	38899	27	56	83	111	138
66	39069	39239	39411	39583	39756	39930	40105	40280	40457	40634	29	58	87	116	145
67	40812	40991	41171	41352	41533	41716	41899	42084	42269	42455	30	61	91	122	152
68	42642	42831	43020	43210	43401	43592	43785	43979	44174	44370	32	64	96	128	160
69	44567	44765	44964	45164	45365	45567	45771	45975	46181	46387	33	68	101	135	169
70	46595	46804	47014	47225	47437	47650	47865	48081	48298	48516	35	71	107	143	178
71	48736	48957	49179	49402	49626	49852	50080	50308	50538	50769	37	75	113	151	189
72	51002	51236	51471	51708	51946	52186	52427	52670	52914	53159	40	80	120	160	200
73	53406	53655	53905	54157	54411	54666	54923	55181	55441	55703	42	85	128	170	213
74	55966	56231	56498	56767	57038	57310	57584	57860	58139	58418	46	91	136	182	228
75	58700	58984	59270	59558	59848	60140	60434	60730	61029	61330	49	98	146	195	244
76	61632	61938	62245	62555	62867	63181	63498	63818	64140	64464	53	105	158	211	263
77	64791	65121	65453	65788	66126	66466	66810	67156	67505	67857	57	114	171	228	285
78	68212	68570	68932	69296	69664	70034	70409	70786	71167	71552	62	124	186	248	311
79	71940	72332	72727	73127	73530	73937	74348	74763	75182	75605	68	136	205	272	341
80	76033	76465	76902	77343	77789	78239	78694	79155	79620	80091	76	151	227	302	378
81	80567	81048	81535	82027	82526	83030	83540	84056	84579	85109	84	169	253	338	423
82	85644	86187	86737	87294	87858	88430	89010	89598	90193	90798	96	192	288	384	480
83	91411	92032	92663	93304	93954	94614	95285	95966	96658	97361	111	222	333	444	556
84	98077	98804	99544	1.00296	1.01063	1.01843	1.02637	1.03447	1.04272	1.05113					
85	1.05970	06846	07739	08651	09583	10536	11510	12506	13526	14571					
86	1.15642	16739	17866	19022	20211	21432	22690	23985	25320	26697					
87	1.28120	29591	31114	32692	34330	36032	37804	39651	41581	43600					
88	1.45718	47945	50292	52778	55406	58208	61204	64422	67897	71676					
89	1.75814	80390	85505	91304	97998	2.05916	2.15607	2.28100	2.45709	2.75812					
90	00														

LOGARITHMS OF COTANGENTS.

Numbers in difference columns to be subtracted, not added.

Degrees	0′ 0°.0	6′ 0°.1	12′ 0°.2	18′ 0°.3	24′ 0°.4	30′ 0°.5	36′ 0°.6	42′ 0°.7	48′ 0°.8	54′ 0°.9	Mean Differences. 1	2	3	4	5
0	∞	2.75812	2.45709	2.28100	2.15606	2.05914	1.97996	1.91300	1.85500	1.80384					
1	1.75808	71668	67888	64410	61191	58193	55389	52755	50271	47921					
2	1.45692	43571	41549	39616	37766	35991	34285	32644	31062	29535					
3	1.28060	26634	25252	23913	22613	21351	20125	18932	17770	16639					
4	1.15536	14460	13409	12384	11381	10402	09443	08505	07586	06687					
5	1.05805	04940	04092	03261	02444	01642	00855	00081	0.99321	0.98573	133	266	398	531	664
6	0.97838	97115	96403	95703	95013	94334	93665	93006	92357	91717	112	225	337	450	563
7	.91086	90463	89850	89224	88647	88057	87475	86901	86333	85773	98	195	293	391	489
8	.85220	84673	84133	83599	83072	82550	82035	81525	81021	80522	86	173	259	346	432
9	.80029	79541	79058	78580	78107	77639	77176	76717	76263	75813	78	155	233	316	388
10	.75368	74927	74490	74057	73628	73203	72782	72365	71951	71541	70	141	212	282	353
11	.71135	70732	70332	69936	69543	69154	68767	68384	68004	67627	64	129	194	258	323
12	.67253	66881	66513	66147	65685	65424	65067	64712	64360	64011	60	119	178	239	298
13	.63664	63319	62977	62637	62300	61965	61632	61301	60973	60647	56	111	167	223	278
14	.60323	60001	59681	59364	59048	58734	58422	58113	57805	57499	52	104	156	208	261
15	.57195	56892	56592	56293	55996	55701	55408	55116	54826	54537	49	98	147	195	245
16	.54250	53965	53681	53399	53119	52840	52562	52286	52011	51738	46	93	139	185	232
17	.51466	51196	50927	50659	50393	50128	49864	49602	49341	49081	44	88	132	176	220
18	.48822	48565	48309	48054	47801	47548	47297	47047	46797	46550	42	84	126	168	210
19	.46303	46053	45813	45569	45327	45085	44845	44605	44367	44130	40	80	120	160	201
20	.43893	43658	43424	43190	42958	42726	42496	42266	42037	41809	38	77	115	154	193
21	.41582	41356	41131	40906	40683	40460	40238	40017	39797	39578	37	74	111	148	185
22	.39359	39141	38924	38708	38492	38278	38064	37850	37638	37426	35	71	107	143	178
23	.37215	37004	36795	36586	36377	36170	35963	35757	35551	35346	34	69	104	138	172
24	.35142	34938	34735	34533	34331	34130	33929	33729	33530	33331	33	66	100	134	167
25	.33133	32935	32738	32542	32346	32150	31956	31761	31568	31374	33	65	98	130	163
26	.31182	30990	30798	30607	30416	30226	30037	29848	29659	29471	32	63	95	127	158
27	.29283	29096	28910	28723	28538	28352	28167	27983	27799	27616	31	62	93	123	154
28	.27433	27250	27068	26886	26705	26524	26343	26163	25983	25804	30	60	90	121	151
29	.25625	25446	25268	25090	24913	24736	24559	24383	24207	24031	29	59	88	118	147
30	.23856	23681	23507	23332	23159	22985	22812	22639	22467	22294	29	58	87	116	144
31	.22123	21951	21780	21609	21438	21268	21098	20928	20759	20590	28	57	85	113	142
32	.20421	20253	20084	19916	19749	19581	19414	19247	19081	18914	28	56	84	112	139
33	.18748	18582	18417	18252	18087	17922	17757	17593	17429	17265	27	55	82	110	137
34	.17101	16938	16775	16612	16449	16287	16124	15962	15800	15639	27	54	81	108	135
35	.15477	15316	15155	14994	14834	14673	14513	14353	14193	14033	27	53	80	107	134
36	.13874	13715	13555	13396	13238	13079	12921	12762	12604	12446	26	53	79	106	132
37	.12289	12131	11973	11816	11659	11502	11345	11188	11032	10875	26	52	79	105	131
38	.10719	10563	10407	10251	10095	09939	09784	09629	09473	09318	26	52	78	104	130
39	.09163	09008	08853	08699	08544	08390	08235	08081	07927	07773	26	51	77	103	129
40	.07619	07465	07311	07157	07004	06850	06697	06543	06390	06237	26	51	77	102	128
41	.06084	05931	05778	05625	05472	05319	05166	05014	04861	04709	25	51	76	102	127
42	.04556	04404	04252	04099	03947	03795	03643	03490	03338	03186	25	51	76	101	127
43	.03034	02882	02731	02579	02427	02275	02123	01971	01820	01668	25	51	76	101	126
44	.01516	01365	01213	01061	00910	00758	00606	00455	00303	00152	25	51	76	101	126

LOGARITHMS OF COTANGENTS

Numbers in difference columns to be subtracted, not added.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	0.000	1.99848	99697	99545	99394	99242	99090	98939	98787	98635	25	51	76	101	126
46	$\bar{1}$.98484	98332	98180	98029	97877	97725	97573	97421	97269	97118	25	51	76	101	120
47	$\bar{1}$.96966	96814	96662	96510	96357	96205	96053	95901	95748	95596	25	51	76	101	127
48	$\bar{1}$.95444	95291	95139	94986	94834	94681	94528	94375	94222	94069	26	51	76	102	127
49	$\bar{1}$.93916	93763	93610	93457	93303	93150	92996	92843	92689	92535	26	51	77	102	128
50	$\bar{1}$.92381	92227	92073	91919	91765	91610	91456	91301	91147	90992	26	51	77	103	129
51	$\bar{1}$.90837	90682	90527	90371	90216	90061	89905	89749	89593	89437	26	52	78	104	130
52	$\bar{1}$.89281	89125	88968	88812	88655	88498	88341	88184	488027	87860	26	52	79	105	131
53	$\bar{1}$.87711	87554	87396	87238	87079	86921	86762	86603	86445	86285	26	53	79	106	132
54	$\bar{1}$.86126	85967	85807	85647	85487	85327	85166	85006	84845	84684	27	53	80	107	134
55	$\bar{1}$.84523	84361	84200	84038	83876	83713	83551	83388	83225	83062	27	54	81	108	135
56	$\bar{1}$.82899	82735	82571	82407	82243	82078	81913	81748	81583	81418	27	55	82	110	137
57	$\bar{1}$.81252	81086	80919	80753	80586	80419	80251	80084	79916	79747	28	56	84	112	139
58	$\bar{1}$.79579	79410	79241	79072	78902	78732	78562	78391	78220	78049	28	57	85	113	142
59	$\bar{1}$.77877	77706	77533	77361	77188	77015	76841	76668	76493	76319	29	58	87	116	144
60	$\bar{1}$.76144	75969	75793	75617	75441	75264	75087	74910	74732	74554	29	59	88	118	147
61	$\bar{1}$.74375	74196	74017	73837	73657	73476	73295	73114	72932	72750	30	60	90	121	151
62	$\bar{1}$.72567	72384	72201	72017	71833	71648	71462	71277	71090	70904	31	62	93	123	154
63	$\bar{1}$.70717	70529	70341	70152	69963	69774	69584	69393	69202	69010	32	63	95	127	158
64	$\bar{1}$.68818	68626	68432	68239	68044	67850	67654	67458	67262	67065	33	65	98	130	163
65	$\bar{1}$.66867	66669	66470	66271	66071	65870	65669	65467	65265	65062	33	67	100	134	169
66	$\bar{1}$.64858	64654	64449	64243	64037	63830	63623	63414	63205	62996	35	69	104	138	173
67	$\bar{1}$.62785	62574	62362	62150	61936	61722	61508	61292	61076	60859	36	71	107	143	179
68	$\bar{1}$.60641	60422	60203	59983	59762	59540	59317	59094	58869	58644	37	74	111	148	185
69	$\bar{1}$.58418	58191	57963	57734	57504	57274	57042	56810	56576	56342	39	77	116	154	193
70	$\bar{1}$.56107	55870	55633	55395	55155	54915	54673	54431	54187	53943	40	80	120	160	201
71	$\bar{1}$.53697	53450	53202	52953	52703	52452	52199	51946	51691	51435	42	84	126	168	210
72	$\bar{1}$.51178	50919	50659	50398	50136	49872	49607	49341	49073	48804	44	88	132	176	220
73	$\bar{1}$.48534	48262	47989	47714	47438	47160	46881	46601	46319	46035	46	93	139	185	232
74	$\bar{1}$.45750	45463	45174	44884	44592	44299	44004	43707	43408	43108	49	98	147	196	244
75	$\bar{1}$.42805	42501	42195	41887	41578	41266	40952	40636	40319	39999	52	104	156	208	261
76	$\bar{1}$.39677	39353	39027	38699	38368	38035	37700	37363	37023	36681	56	111	167	223	278
77	$\bar{1}$.36336	35989	35640	35288	34933	34576	34215	33853	33487	33119	60	119	179	239	299
78	$\bar{1}$.32747	32373	31996	31616	31233	30846	30457	30064	29668	29268	64	129	194	258	323
79	$\bar{1}$.28865	28459	28049	27635	27218	26797	26372	25943	25510	25073	70	141	212	282	353
80	$\bar{1}$.24632	24186	23737	23283	22824	22361	21893	21420	20942	20459	78	155	233	311	388
81	$\bar{1}$.19971	19478	18979	18475	17965	17450	16928	16401	15867	15327	86	173	259	346	432
82	$\bar{1}$.14780	14227	13667	13099	12525	11943	11353	10756	10150	09537	98	195	293	391	489
83	$\bar{1}$.08914	08283	07643	06994	06335	05666	04987	04297	03597	02885	112	225	337	450	562
84	$\bar{1}$.02162	$\bar{1}$.01427	$\bar{1}$.00679	$\bar{2}$.99919	$\bar{2}$.99145	$\bar{2}$.98358	$\bar{2}$.97556	$\bar{2}$.96739	$\bar{2}$.95908	$\bar{2}$.95060	133	266	398	531	664
85	$\bar{2}$.94195	93313	92414	91495	90557	89598	88618	87616	86591	85540					
86	$\bar{2}$.84464	83361	82230	81068	79875	78649	77387	76087	74748	73366					
87	$\bar{2}$.71640	70465	68938	67356	65715	64009	62234	60384	58451	56429					
88	$\bar{2}$.54308	52079	49792	47245	44611	41807	38809	35590	32112	28332					
89	$\bar{2}$.24192	$\bar{2}$.19616	$\bar{2}$.14500	$\bar{2}$.08700	$\bar{2}$.02004	$\bar{3}$.94086	$\bar{3}$.84394	$\bar{3}$.71900	$\bar{3}$.54291	$\bar{3}$.24188					

DEGRESS TO RADIANS.

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences.				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
0	00000	00175	00349	00524	00698	00873	01047	01222	01396	01571	29	58	87	116	145
1	.01745	01920	02094	02269	02443	02618	02793	02967	03142	03316	29	58	87	116	145
2	.03491	03665	03840	04014	04189	04363	04538	04712	04887	05061	29	58	87	116	145
3	.05236	05411	05585	05760	05934	06109	06283	06458	06632	06807	29	58	87	116	145
4	.06981	07156	07330	07505	07679	07854	08029	08203	08378	08552	29	58	87	116	145
5	.08727	08901	09076	09250	09425	09599	09774	09948	10123	10297	29	58	87	116	145
6	.10472	10647	10821	10996	11170	11345	11519	11694	11868	12043	29	58	87	116	145
7	.12217	12392	12566	12741	12915	13090	13264	13439	13614	13788	29	58	87	116	145
8	.13963	14137	14312	14486	14661	14835	15010	15184	15359	15533	29	58	87	116	145
9	.15708	15882	16057	16232	16406	16581	16755	16930	17104	17279	29	58	87	116	145
10	.17453	17628	17802	17977	18151	18326	18500	18675	18850	19024	29	58	87	116	145
11	.19199	19373	19548	19722	19897	20071	20246	20420	20595	20769	29	58	87	116	145
12	.20944	21118	21293	21468	21642	21817	21991	22166	22340	22515	29	58	87	116	145
13	.22689	22864	23038	23213	23387	23562	23736	23911	24086	24260	29	58	87	116	145
14	.24435	24609	24784	24958	25133	25307	25482	25656	25831	26005	29	58	87	116	145
15	.26180	26354	26529	26704	26878	27053	27227	27402	27576	27751	29	58	87	116	145
16	.27925	28100	28274	28449	28623	28798	28972	29147	29322	29496	29	58	87	116	145
17	.29671	29845	30020	30194	30369	30543	30718	30892	31067	31241	29	58	87	116	145
18	.31416	31590	31765	31940	32114	32289	32463	32638	32812	32987	29	58	87	116	145
19	.33161	33336	33510	33685	33859	34034	34208	34383	34558	34732	29	58	87	116	145
20	.34907	35081	35256	35430	35605	35779	35954	36128	36303	36477	29	58	87	116	145
21	.36652	36826	37001	37176	37350	37525	37699	37874	38048	38223	29	58	87	116	145
22	.38397	38572	38746	38921	39095	39270	39444	39619	39794	39968	29	58	87	116	145
23	.40143	40317	40492	40666	40841	41015	41190	41364	41539	41713	29	58	87	116	145
24	4.1888	42062	42237	42411	42586	42761	42935	43110	43284	43459	29	58	87	116	145
25	.43633	43808	43982	44157	44331	44506	44680	44855	45029	45204	29	58	87	116	145
26	.45379	45553	45728	45902	46077	46251	46426	46600	46775	46949	29	58	87	116	145
27	.47124	47298	47473	47647	47822	47997	48171	48346	48520	48695	29	58	87	116	145
28	.48869	49044	49218	49393	49567	49742	49916	50091	50265	50440	29	58	87	116	145
29	.50615	50789	50964	51138	51313	51487	51662	51836	52011	52185	29	58	87	116	145
30	.52360	52534	52709	52883	53058	53233	53407	53582	53756	53931	29	58	87	116	145
31	.54105	54280	54454	54629	54803	54978	55152	55327	55501	55676	29	58	87	116	145
32	.55851	56025	56200	56374	56549	56723	56898	57072	57247	57421	29	58	87	116	145
33	.57596	57770	57945	58119	58294	58469	58643	58818	58992	59167	29	58	87	116	145
34	.59341	59516	59690	59865	60039	60214	60388	60563	60737	60912	29	58	87	116	145
35	.61087	61261	61436	61610	61785	61959	62134	62308	62483	62657	29	58	87	116	145
36	.62832	63006	63181	63355	63530	63705	63879	64054	64228	64403	29	58	87	116	145
37	.64577	64752	64926	65101	65275	65450	65624	65799	65973	66148	29	58	87	116	145
38	.66323	66497	66672	66846	67021	67195	67370	67544	67719	67893	29	58	87	116	145
39	.68068	68242	68417	68591	68766	68941	69115	69290	69464	69639	29	58	87	116	145
40	.69813	69988	70162	70337	70511	70686	70860	71035	71209	71384	29	58	87	116	145
41	.71558	71733	71908	72082	72257	72431	72606	72780	72955	73129	29	58	87	116	145
42	.73304	73478	73653	73827	74002	74176	74351	74526	74700	74875	29	58	87	116	145
43	.75049	75224	75398	75573	75747	75922	76096	76271	76445	76620	29	58	87	116	145
44	.76794	76969	77144	77318	77493	77667	77842	78016	78191	78365	29	58	87	116	145

DEGRESS TO RADIANS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences				
	0°.0	0°.1	0°.2	0°.3	0°.4	0°.5	0°.6	0°.7	0°.8	0°.9	1	2	3	4	5
45	.78540	78714	78889	79063	79238	79412	79587	79762	79936	80111	29	58	87	116	145
46	.80285	80460	80634	80809	80983	81158	81332	81507	81681	81856	29	58	87	116	145
47	.82030	82205	82380	82554	82729	82903	83078	83252	83427	83601	29	58	87	116	145
48	.83776	83950	84125	84299	84474	84648	84823	84998	85172	85347	29	58	87	116	145
49	.85521	85696	85870	86045	86219	86394	86568	86743	86917	87092	29	58	87	116	145
50	.87266	87441	87616	87790	87965	88139	88314	88488	88663	88837	29	58	87	116	145
51	.89012	89186	89361	89535	89710	89884	90059	90234	90408	90583	29	58	87	116	145
52	.90757	90932	91106	91281	91455	91630	91804	91979	92153	92328	29	58	87	116	145
53	.92502	92677	92852	93026	93201	93375	93550	93724	93899	94073	29	58	87	116	145
54	.94248	94422	94597	94771	94946	95120	95295	95470	95644	95819	29	58	87	116	145
55	.95993	96168	96342	96517	96691	96866	97040	97215	97389	97564	29	58	87	116	145
56	.97484	97913	98088	98262	98437	98611	98786	98960	99135	99309	29	58	87	116	145
57	.99848	99658	99833	1.00007	1.00182	1.00356	1.00531	1.00706	1.00880	1.01055	29	58	87	116	145
58	1.01229	01404	01578	01753	01927	02102	02276	02451	02625	02800	29	58	87	116	145
59	1.02974	03149	03323	03498	03673	03847	04022	04196	04371	04545	29	58	87	116	145
60	1.04720	04894	05069	05243	05418	05592	05767	05941	06116	06291	29	58	87	116	145
61	1.06465	06640	06814	06989	07163	07338	07512	07687	07861	08036	29	58	87	116	145
62	1.08210	08385	08559	08734	08909	09083	09258	09432	09607	09781	29	58	87	116	145
63	1.09956	10130	10305	10479	10654	10828	11003	11177	11352	11527	29	58	87	116	145
64	1.11701	11876	12050	12225	12399	12574	12748	12923	13097	13272	29	58	87	116	145
65	1.13446	13621	13795	13970	14145	14319	14494	14668	14843	15017	29	58	87	116	145
66	1.15192	15366	15541	15715	15890	16064	16239	16413	16588	16763	29	58	87	116	145
67	1.16937	17112	17286	17461	17635	17810	17984	18159	18333	18508	29	58	87	116	145
68	1.18682	18857	19031	19602	19381	19555	19730	19904	20079	20253	29	58	87	116	145
69	1.20428	20602	20777	20951	21126	21300	21475	21649	21824	21999	29	58	87	116	145
70	1.22173	22348	22522	22697	22871	23046	23220	23395	23569	23744	29	58	87	116	145
71	1.23918	24093	24267	24442	24617	24791	24966	25140	25315	25489	29	58	87	116	145
72	1.25664	25838	26013	26187	26362	26536	26711	26885	27060	27234	29	58	87	116	145
73	1.27409	27584	27758	27933	28107	28282	28456	28631	28805	28980	29	58	87	116	145
74	1.29154	29329	29503	29678	29852	30027	30202	30376	30551	30725	29	58	87	116	145
75	1.30900	31074	31249	31423	31598	31772	31947	32121	32296	32470	29	58	87	116	145
76	1.32645	32820	32994	33169	33343	33518	33692	33867	34041	34216	29	58	87	116	145
77	1.34390	34565	34739	34914	35088	35263	35438	35612	35787	35961	29	58	87	116	145
78	1.36136	36310	36485	36659	36834	37008	37183	37357	37532	37706	29	58	87	116	145
79	1.37881	38056	38230	38405	38579	38754	38928	39103	39277	39452	29	58	87	116	145
80	1.39626	39801	39975	40150	40324	40499	40674	40848	41023	41197	29	58	87	116	145
81	1.14372	41545	41721	41895	42070	42244	42419	42593	42768	42942	29	58	87	116	145
82	1.43117	43292	43466	43641	43815	43990	44164	44339	44513	44688	29	58	87	116	145
83	1.44862	45037	45211	45386	45560	45735	45910	46084	46259	46433	29	58	87	116	145
84	1.46608	46782	46957	47131	47306	47480	47655	47829	48004	48178	29	58	87	116	145
85	1.48353	48528	48702	48877	49051	49226	49400	49575	49749	49924	29	58	87	116	145
86	1.50098	50273	50447	50622	50796	50971	51146	51320	51495	51669	29	58	87	116	145
87	1.31844	52018	52193	52367	52542	52716	52891	53065	53240	53414	29	58	87	116	145
88	1.53589	53764	53938	54113	54287	54462	54636	54811	54985	55160	29	58	87	116	145
89	1.55334	55509	55683	55858	56032	56207	56381	56556	56731	56905	29	58	87	116	145

CONSTANTS

One Radian = 57° 17′ 45″ nearly = 206265^{4} ;
log 206265 = 5·3144255

$\pi = 3 \cdot 1419265.$ $\qquad \log \pi = 0 \cdot 4971499.$

$\frac{1}{\pi} = 0 \cdot 31830989.$ $\qquad \log \frac{\mathrm{I}}{\pi} = \bar{1} \cdot 5028501.$

$\frac{\pi}{180} = 0 \cdot 01745329.$ $\qquad \log \frac{\pi}{180} = \bar{2} \cdot 2418774.$

$\frac{180}{\pi} = 57 \cdot 2957795.$ $\qquad \log \frac{180}{\pi} = 1 \cdot 7581226$

$\pi^2 = 9 \cdot 86960440.$ $\qquad \log \pi^2 = 0 \cdot 9942997.$

$\frac{1}{\pi^2} = 0 \cdot 10132118.$ $\qquad \log \frac{1}{\pi^2} = \bar{1} \cdot 0057003.$

$\sqrt{\pi} = 1 \cdot 77245385.$ $\qquad \log \sqrt{\pi} = 0 \cdot 2485749.$

$\frac{1}{\sqrt{\pi}} = 0 \cdot 56418958.$ $\qquad \log \frac{\mathrm{I}}{\sqrt{\pi}} = \bar{1} \cdot 7514251.$

$\sqrt[3]{\pi} = 1 \cdot 46459189.$ $\qquad \log \sqrt[3]{\pi} = 0 \cdot 1657166.$

$\frac{\mathrm{I}}{\sqrt[3]{\pi}} = 0 \cdot 68278406.$ $\qquad \log \frac{\mathrm{I}}{\sqrt[3]{\pi}} = \bar{1} \cdot 8342834.$

$\sqrt{2} = 1 \cdot 4142135...$ $\qquad \sqrt{3} = 1 \cdot 7320508...$

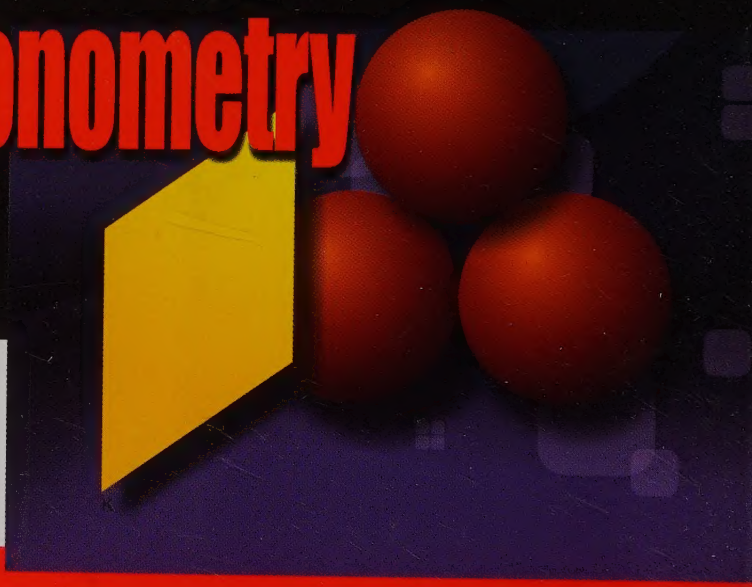
$\sqrt{5} = 2 \cdot 23606789...$ $\qquad \sqrt{6} = 2 \cdot 4494897...$

$\sqrt{7} = 2 \cdot 6457513...$ $\qquad \sqrt{8} = 2 \cdot 8284271...$

$\sqrt{10} = 3 \cdot 162277...$

Success cannot be assured .....
it has to be attain

PLANE
Trigonometry



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